Homework Assignment 2.

Latest due date: midnight - Sunday February 2nd
Possible bonus of 0.1 if 75% correct (≥ 21 pts.)

For full credit: answer all questions fully by providing all details of what you are doing and why. All relevant equations must be included. **Explanations must be extensive!**

Please remember that your report must be anonymous! Peer Reviews due by Tuesday midnight (latest)

1. (4 pts.) Consider the function \( f(x, y) = xe^y \) and produce (by hand) its quadratic polynomial approximation to be used to estimate \( f(0.1, 0.1) \) using a Taylor expansion. What is the approximation and the corresponding theoretical error? What is the true error?

2. (4 pts.) Program Newton’s method for the system of equations below. Starting from the initial value of \( x = (1, 2, 3) \) produce a table with the corresponding approximation for 10 iterations. Also include the true error in the last column of your table for each iteration.

\[
\begin{align*}
  f_1(x) &= x_1^2 - 2x_1 + x_2^2 - x_3 + 1 = 0 \\
  f_2(x) &= x_1x_2^2 - x_1 - 3x_2 + x_2x_3 + 2 = 0 \\
  f_3(x) &= x_1x_2^3 - 3x_3 + x_2x_3^2 + x_1x_2 = 0
\end{align*}
\]

3. (4 pts.) Suppose you need to solve the equation \( A^2x = b \), where \( A \) is nxn and invertible. Which of the following options is likely to take the least computer time?

   - Use \( LU \) factorization - also suggest how you would apply it.
   - Compute \( A^{-1} \) first, followed by \( x = A^{-1}(A^{-1}b) \).

Substantiate your answer in your report for each case and provide an approximate operations count.

4. (4 pts.) The age distribution in a population of female beetles can be modeled with the matrix \( A \) provided below which shows survival rates on a year by year basis. Let \( x_k \) denote the distribution of ages in year \( k \), with \( x_{k,1} \) representing the number of beetles of age one in year \( k \), and so on. Then the relation between populations in successive years \( k \) can be expressed as \( x_{k+1} = Ax_k \) where

\[
A = \begin{pmatrix}
  0 & 0 & 6 \\
  1/3 & 0 & 0 \\
  0 & 1/2 & 0
\end{pmatrix}.
\]

If after five years the beetle population has the distribution \((300, 60, 30)\), what was the original distribution? Outline the methodology you used in your report as well.

5. (4 pts.) (a) On paper (no computer), solve the system of equations \( Ax = b \) below using permuted \( LU \) factorization showing all your steps. Write down your solution \( x \) and the matrices \( L \) and \( U \).

\[
\begin{bmatrix}
  3 & 1 & 2 \\
  6 & 1 & 3 \\
  6 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
= 
\begin{bmatrix}
  1 \\
  1 \\
  1
\end{bmatrix}.
\]

(b) Exactly how many operations did it take to produce the solution \( x \)? Does that number agree with theory?

6. (4 pts.) Perform, by hand, 2 iterations of the Jacobi iterative scheme for \( Ax = b \) from above and compute an approximation for \( x \) starting from an initial value of \( x_0 = [0, 1, 1]^T \).

(Note that \([0, 1, 1]^T\) is just notation and denotes that \( x_0 \) is really a column vector).

7. (4 pts.) Repeat the task above for the Gauss-Seidel iterative scheme.