1.3 Group theory 1: The classical matrix groups

Consider the set of all complex $N \times N$ matrices $U$ satisfying the unitarity condition

\[ U^\dagger = U^{-1}, \quad \text{or} \quad U^\dagger U = UU^\dagger = 1, \quad (1.5) \]

leaving the scalar product in $\mathbb{C}^N$ invariant: If $\chi, \chi' \in \mathbb{C}^N$ then $\chi'^\dagger \chi = \chi^\dagger \chi$. The unitarity property survives matrix multiplication which means that if two matrices $U_1$ and $U_2$ are unitary so is the product $U_3 = U_1 U_2$ which hence also belongs to this matrix set:

\[ U_3^\dagger U_3 = (U_1 U_2)^\dagger U_1 U_2 = U_2^\dagger U_1^\dagger U_1 U_2 = 1. \quad (1.6) \]

Note that the unit matrix $1$ is in this set of matrices and that every matrix has an inverse, which is actually part of the assumption above (since $U^\dagger = U^{-1}$ implies that $\det U = e^{i\alpha}$ with $\alpha \in \mathbb{R}$). We may also add the requirement that the matrices have unit determinant which is also preserved by matrix multiplication since $\det (U_1 U_2) = (\det U_1)(\det U_2)$. This is a subset of the previous one without the unit determinant condition.

If we consider all matrices in either one of these two sets the following properties are trivially satisfied:

1. The set is closed under multiplication (here matrix multiplication)
2. The multiplication is associative (as matrix multiplication always is)
3. There is a unit (here the unit matrix $1$)
4. Every element in the set has an inverse (true here since we consider only matrices satisfying $U^\dagger = U^{-1}$, i.e., matrices with non-zero determinant

Viewing these properties instead as axioms they define a group, often denoted $G$ and members of the set are called elements, which in the case above is called the unitary group:

\[ U(N), \quad (1.7) \]

realised in this discussion in terms of complex $N \times N$ matrices. When the condition that the matrices have unit determinant $\det U = 1$ is added the group is called special unitary:

\[ SU(N). \quad (1.8) \]

The standard model of elementary particles is based on three such groups: $U(1)$, $SU(2)$ and $SU(3)$. The group $U(1)$ is the group of phases, i.e., multiplication by $e^{i\alpha}$ where $\alpha$ is a (real) parameter (angle), and is abelian. The other two groups are non-abelian, i.e., $g_1 g_2 \neq g_2 g_1$ for some $g_1, g_2 \in G$. Any $U \in SU(2)$ can be written in terms of $a, b \in \mathbb{C}$ as

\[ U \in SU(2) : \quad U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad \text{where} \quad |a|^2 + |b|^2 = 1, \quad (1.9) \]

which is easily checked. Two general such matrices do not commute and the group is therefore non-abelian. The set of matrices in this group can be parametrised by the points
on the unit three-sphere $S^3$ since $|a|^2 + |b|^2 = 1$ which, if written in real variables $a = x_1 + ix_2$ and $b = x_3 + ix_4$, becomes $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, i.e., the equation for $S^3$ embedded in $\mathbb{R}^4$.

While $U(1)$ and $SU(2)$ can be viewed in this way in terms of simple geometries $S^1$ (unit complex numbers) and $S^3$ (unit quaternions) $SU(3)$ and higher $SU(N)$ groups are much more complicated as manifolds and do not correspond to anything with other names.

These notions can be carried over to real matrices leading to the orthogonal groups

$$O(N) \text{ or } SO(N).$$

(1.10)

Among the groups called classical there is just one other case, the symplectic groups. One way to define matrix groups is by looking for matrices preserving some special matrices numerically. One example is the unit matrix which then leads to orthogonal groups ($g1g^T = 1 \Rightarrow g \in O(N)$) while invariance of the antisymmetric $2N \times 2N$ matrix

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(1.11)

defines the symplectic groups

$$Sp(2N).$$

(1.12)

Comments:

1. The complete classification of all finite-dimensional Lie groups (i.e., continuous of the kind discussed above) is known as the Cartan classification and contains in addition to the classical groups above, i.e., the unitary (denoted $A_n$ by Cartan), orthogonal (denoted $B_n$ or $D_n$), and symplectic (denoted $C_n$), also the exceptional ones $G_2, F_4, E_6, E_7, E_8$. These latter ones have, however, no simple definition in terms of matrices similar to the one used above. The index $n$ (and the indices appearing on the exceptional groups) is the rank of the group related to the maximal number of matrices that can be diagonalised simultaneously. The classes $A_n$ contain also the groups $GL(N)$ and $SL(N)$ which are general matrices with non-zero or unit determinant, respectively.

2. There are also other important groups like finite ones with a finite number of group elements, and those with a discrete set of elements which is infinite in number (see courses in Group theory and in String theory).

3. Other important Lie groups in physics have infinite dimension. Examples of such are Virasoro and Kac-Moody appearing in two-dimensional conformal field theory ($CFT_2$) used, e.g., in string theory and in the context of phase transitions in condensed matter systems.

4. There is a very important (also for QFT) distinction between compact groups (e.g., $U(N), SU(N), SO(N)$) and non-compact ones (e.g., $SO(1, 3), SU(1, 1), Sp(2N), SL(N)$) discussed further in "Group theory 2" on Lie algebras and representations.

There is a third case, the octonions, for which a unit octonion can be shown to be the same as the seven-dimensional sphere, $S^7$. This manifold is, however, not a group manifold but does nevertheless share some properties with group manifolds as, e.g., being parallelisable. $S^7$ plays a key role in string/M theory.

Note that some authors call these groups $Sp(N)$.