FOURIER ANALYSIS & METHODS 2020.03.06

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. The Legendre polynomials and applications

Theorem 1. *The Legendre polynomials are orthogonal in $\mathcal{L}^2(-1,1)$, and*

$$||P_n||^2 = \frac{2}{2n+1}.$$

Proof: We first prove the orthogonality. Assume that $n > m$. Then, since they have this constant stuff out front, we compute

$$2^n n! 2^m m! \langle P_n, P_m \rangle = \int_{-1}^1 \frac{d^n}{dx^n}(x^2-1)^n \frac{d^m}{dx^m}(x^2-1)^m dx.$$

Let us integrate by parts once:

$$= \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^m}{dx^m}(x^2-1)^m \Big|_{-1}^1 - \int_{-1}^1 \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m.$$

Consider the boundary term:

$$\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n = \frac{d^{n-1}}{dx^{n-1}}(x-1)^n(x+1)^n.$$

This vanishes at $x = \pm 1$, because the polynomial vanishes to order n whereas we only differentiate $n-1$ times. So, we have shown that

$$2^n n! 2^m m! \langle P_n, P_m \rangle = -\int_{-1}^1 \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m.$$

We repeat this $n-1$ more times. We note that for all $j < n$,

$$\frac{d^j}{dx^j}(x^2-1)^n \text{ vanishes at } x = \pm 1.$$

For this reason, all of the boundary terms from integrating by parts vanish. So, we just get

$$(-1)^n \int_{-1}^1 (x^2-1)^n \frac{d^{m+n}}{dx^{m+n}}(x^2-1)^m dx = (-1)^n \int_{-1}^1 (x^2-1)^n \frac{d^n}{dx^n}\frac{d^m}{dx^m}(x^2-1)^m dx$$

Remember that $n > m$. We computed that $\frac{d^m}{dx^m}(x^2-1)^m$ is a polynomial of degree m. So, if we differentiate it more than m times we get zero. So, we're integrating zero! Hence it is zero.

For the second part, we need to compute:

$$(x^2-1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (x^2)^k = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} x^{2k}.$$

Therefore, if we differentiate n times, only the terms with $k \geq n/2$ survive. Differentiating a term x^{2k} once we get $2kx^{2k-1}$. Differentiating n times gives

$$\frac{d^n}{dx^n}(x^{2k}) = x^{2k-n} \prod_{j=0}^{n-1} (2k-j).$$

If we want to be really persnickety, we prove this by induction. For $n=1$, we get that

$$(x^{2k})' = 2kx^{2k-1}.$$

Which is correct. If we assume the formula is true for n, then differentiating $n+1$ times using the formula for n we get

$$(2k-n)x^{2k-(n+1)} \prod_{j=0}^{n-1} (2k-j) = x^{2k-(n+1)} \prod_{j=0}^{n} (2k-j).$$

See, it is correct. As a result,

$$P_n(x) = \frac{1}{2^n n!} \sum_{k \geq n/2}^n (-1)^{n-k} \binom{n}{k} x^{2k-n} \prod_{j=0}^{n-1} (2k-j).$$

So, we see that this is indeed a polynomial of degree n. With this formula, we can write

$$P_n(x) = \frac{1}{2^n n!} \sum_{k \geq n/2}^n (-1)^{n-k} \binom{n}{k} x^{2k-n} \prod_{j=0}^{n-1} (2k-j).$$

Differentiating n times gives us just the term with the highest power of x, so we have

$$\frac{d^n}{dx^n} P_n(x) = \frac{1}{2^n n!} n! \prod_{j=0}^{n-1} (2n-j) = \frac{(2n)!}{2^n n!}.$$

Consequently,

$$\langle P_n, P_n \rangle = (-1)^n \frac{1}{2^n n!} \frac{(2n)!}{2^n n!} \int_{-1}^1 (x^2-1)^n dx = (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (x^2-1)^n dx$$

$$= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} x^{2k} dx$$

$$= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^n (-1)^{n-k} \frac{x^{2k+1}}{2k+1} \binom{n}{k} \Bigg|_0^1$$

$$= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \frac{1}{2k+1}$$

$$= \frac{2(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{2k+1}.$$

This looks super complicated. Apparently by some miracle of life

$$\int_0^1 (1-x^2)^n dx = \frac{\Gamma(n+1)\Gamma(1/2)}{\Gamma(n+3/2)}.$$

Since

$$\langle P_n, P_n \rangle = (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (x^2-1)^n dx = \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (1-x^2)^n dx,$$

we get

$$\frac{\Gamma(n+1)\Gamma(1/2)2(2n)!}{2^{2n}(n!)^2\Gamma(n+3/2)}.$$

We use the properties of the Γ function together with the fact that $\Gamma(1/2) = \sqrt{\pi}$ to obtain

$$\frac{\sqrt{\pi}2(2n)!}{2^{2n}n!(n+1/2)\Gamma(n+1/2)}.$$

Let us consider

$$2(n+1/2)\Gamma(n+1/2) = (2n+1)\Gamma(n+1/2).$$

Next consider

$$2(n-1/2)\Gamma(n-1/2) = (2n-1)\Gamma(n-1/2).$$

Proceeding this way, the denominator becomes

$$2^n n!(2n+1)(2n-1)\dots 1\sqrt{\pi}.$$

However, now looking at the first part

$$2^n n! = 2n(2n-2)(2n-4)\dots 2.$$

So together we get

$$(2n+1)!\sqrt{\pi}.$$

Hence putting this in the denominator of the expression we had above, we have

$$\frac{\sqrt{\pi}2(2n)!}{(2n+1)!\sqrt{\pi}} = \frac{2}{2n+1}.$$

Corollary 2. *The Legendre polynomials are an orthogonal basis for $\mathcal{L}^2$ on the interval* $[-1,1]$.

Theorem 3. *The even degree Legendre polynomials $\{P_{2n}\}_{n\in\mathbb{N}}$ are an orthogonal basis for $\mathcal{L}^2(0,1)$. The odd degree Legendre polynomials $\{P_{2n+1}\}_{n\in\mathbb{N}}$ are an orthogonal basis for $\mathcal{L}^2(0,1)$.*

Proof: Let f be defined on $[0,1]$. We can extend f to $[-1,1]$ either evenly or oddly. First, assume we have extended f evenly. Then, since $f \in \mathcal{L}^2$ on $[0,1]$,

$$\int_{-1}^1 |f_e(x)|^2 dx = 2\int_0^1 |f(x)|^2 dx < \infty.$$

Therefore f_e is in $\mathcal{L}^2$ on the interval $[-1,1]$. We have proven that the Legendre polynomials are an orthogonal basis. So, we can expand f_e in a Legendre polynomial series, as

$$\sum_{n\geq 0} \hat{f}_e(n) P_n,$$

where

$$\hat{f}_e(n) = \frac{\langle f_e, P_n \rangle}{||P_n||^2.}$$

By definition,

$$\langle f_e, P_n \rangle = \int_{-1}^{1} f_e(x) P_n(x) dx.$$

Since f_e is even, the product $f_e(x)P_n(x)$ is an *odd* function whenever n is odd. Hence all of the odd coefficients vanish. Moreover,

$$\langle f_e, P_{2n} \rangle = 2 \int_0^1 f(x) P_{2n}(x)) dx.$$

We also have

$$||P_{2n}||^2 = 2 \int_0^1 |P_{2n}(x)|^2 dx.$$

Consequently

$$f = \sum_{n \in \mathbb{N}} \left(\frac{\int_0^1 f(x) P_{2n}(x) dx}{\int_0^1 |P_{2n}(x)|^2 dx} \right) P_{2n}.$$

We can also extend f oddly. This odd extension satisfies

$$\int_{-1}^{1} |f_o(x)|^2 dx = \int_{-1}^{0} |f_o(x)|^2 dx + \int_0^1 |f_o(x)|^2 dx = 2 \int_0^1 |f_o(x)|^2 dx < \infty.$$

So, the odd extension is also in $\mathcal{L}^2$ on the interval $[-1, 1]$. We can expand f_o in a Legendre polynomial series, as

$$\sum_{n \geq 0} \hat{f}_o(n) P_n,$$

where

$$\hat{f}_o(n) = \frac{\langle f_o, P_n \rangle}{||P_n||^2.}$$

By definition,

$$\langle f_o, P_n \rangle = \int_{-1}^{1} f_o(x) P_n(x) dx.$$

Since f_o is odd, the product $f_o(x)P_n(x)$ is an *odd* function whenever n is *even.* Hence all of the even coefficients vanish. Moreover,

$$\langle f_o, P_{2n+1} \rangle = 2 \int_0^1 f(x) P_{2n+1}(x)) dx,$$

because the product of two odd functions is an even function. We also have

$$||P_{2n+1}||^2 = \int_{-1}^{0} |P_{2n+1}(x)|^2 dx + \int_0^1 |P_{2n+1}(x)|^2 dx = 2 \int_0^1 |P_{2n+1}(x)|^2 dx.$$

Consequently

$$f = \sum_{n \in \mathbb{N}} \left(\frac{\int_0^1 f(x) P_{2n+1}(x) dx}{\int_0^1 |P_{2n+1}(x)|^2 dx} \right) P_{2n+1}.$$

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1.1. **Applications of Legendre polynomials to best approximations on bounded integrals.**

Exercise 1. *Find the polynomial $q(x)$ of at most degree 10 which minimizes the following integral*

$$\int_{-\pi}^{\pi} |q(x) - \sin(x)|^2 dx.$$

To do this exercise, we need different polynomials... If Legendre polynomials are orthogonal on $(-1, 1)$, can we somehow use them to create orthogonal polynomials on $(-\pi, \pi)$? Let's think about changing variables. How about setting

$$t = \frac{x}{\pi}.$$

Then,

$$\int_{-\pi}^{\pi} P_n(x/\pi)\overline{P_m(x/\pi)}dx = \int_{-1}^{1} P_n(t)\overline{P_m(t)}\pi dt = \begin{cases} 0 & n \neq m \\ \frac{2\pi}{2n+1} & n = m \end{cases}.$$

Therefore the polynomials

$$P_n(x/\pi)$$

are orthogonal on $x \in (-\pi, \pi)$, and their norms squared on that interval are

$$\frac{2\pi}{2n+1}.$$

The best approximation is therefore the polynomial

$$q(x) = \sum_{n=0}^{10} a_n P_n(x/\pi), \quad a_n := \frac{\int_{-\pi}^{\pi} \sin(x)\overline{P_n(x/\pi)}dx}{\frac{2\pi}{2n+1}}.$$

Exercise 2. *Find the polynomial $p(x)$ of degree at most 100 which minimizes the following integral*

$$\int_0^{10} |e^{x^2} - p(x)|^2 dx.$$

Yikes! Well, let's not panic just yet. The number 100 is even. Hence, we know that the even degree Legendre polynomials are an orthogonal basis for $\mathcal{L}^2(0, 1)$. So, we can use the even degree Legendre polynomials if we can just deal with this interval not being $(0, 1)$ but being $(0, 10)$. To figure this out, let's think about changing variables... As before, think about changing variables,

$$t = x/10,$$

so that

$$\int_0^{10} P_{2n}(x/10)P_{2m}(x/10)dx = \int_0^1 P_{2n}(t)P_{2m}(t)10dt = \begin{cases} 0 & n \neq m \\ \frac{10}{4n+1} & n = m \end{cases}$$

The last calculation we obtained by recalling our calculation

$$\int_{-1}^{1} |P_n(x)|^2 dx = (-1)^n \frac{(2n)!}{(2^n n!)^2} \int_{-1}^{1} (x^2-1)^n dx = \frac{2}{2n+1} \implies \int_0^1 |P_{2n}(x)|^2 dx = \frac{1}{4n+1}.$$

So, the functions $P_{2n}(x/10)$ are an orthogonal basis for $\mathcal{L}^2(0,10)$. Consequently the Best Approximation Theorem says that the best approximation is given by the polynomial

$$p(x) = \sum_{n=0}^{50} c_n P_{2n}(x/10), \quad c_n = \frac{\int_0^{10} e^{x^2} \overline{P_{2n}(x/10)} dx}{\frac{10}{4n+1}}.$$

Exercise 3. *Find the polynomial $p(x)$ of degree at most 99 which minimizes the following integral*

$$\int_0^{10} |e^{x^2} - p(x)|^2 dx.$$

Here, we can recycle our previous solution since 99 is odd, so we can use the odd degree Legendre polynomials in this case to form an orthogonal basis for $\mathcal{L}^2(0,10)$. Our polynomial shall be

$$p(x) = \sum_{n=0}^{49} c_n P_{2n+1}(x/10), \quad c_n = \frac{\int_0^{10} e^{x^2} \overline{P_{2n+1}(x/10)} dx}{\frac{10}{2(2n+1)+1}}.$$

1.2. **Legendre polynomials for best approximations on arbitrary intervals.** Let's consider a best approximation problem on an interval (a,b). First, we find its midpoint,

$$m = \frac{a+b}{2}.$$

Next, we find its length

$$\ell = \frac{b-a}{2}.$$

Then the interval

$$(a,b) = (m-\ell, m+\ell).$$

Since we know about the Legendre polynomials, P_n, on $(-1,1)$ since $x \mapsto \frac{x-m}{\ell} = t$ sends (a,b) to $(-1,1)$,

$$P_n\left(\frac{x-m}{\ell}\right) \qquad \text{are orthogonal on } (a,b).$$

In case this is not super obvious, let us compute using the substitution $t = \frac{x-m}{\ell}$,

$$\int_a^b P_n\left(\frac{x-m}{\ell}\right) P_k\left(\frac{x-m}{\ell}\right) dx = \int_{-1}^1 \ell P_n(t) P_k(t) dt = 0 \text{ if } n \neq k.$$

We have simply used substitution in the integral with $t = \frac{x-m}{\ell}$. So, these modified Legendre polynomials are orthogonal on (a,b). Moreover

$$\int_a^b P_n^2\left(\frac{x-m}{\ell}\right) dx = \int_{-1}^1 \ell P_n^2(t) dt = \ell ||P_n||^2 = \frac{2\ell}{2n+1}.$$

So, we simply expand the function f using this version of the Legendre polynomials. Let

$$c_n = \frac{\int_a^b f(x) P_n\left(\frac{x-m}{\ell}\right) dx}{\int_a^b [P_n((x-m)/\ell)]^2 dx}.$$

The best approximation amongst all polynomials of degree at most N is therefore

$$P(x) = \sum_{n=0}^{N} c_n P_n\left(\frac{x-m}{\ell}\right).$$

2. Les polynomes d'hermite

These polynomials shall be a basis for $\mathcal{L}^2(\mathbb{R})$ with respect to the weight function e^{-x^2}.

Definition 4. *The Hermite polynomials are defined to be*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Proposition 5. *The Hermite polynomials are polynomials with the degree of H_n equal to n.*

Proof: The proof is by induction. For $n = 0$, this is certainly true, as $H_0 = 1$. Next, let us assume that

$$\frac{d^n}{dx^n} e^{-x^2} = p_n(x) e^{-x^2},$$

is true for a polynomial, p_n which is of degree n. Then,

$$\frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = \frac{d}{dx}\left(p_n(x)e^{-x^2}\right) = p_n'(x)e^{-x^2} - 2xp_n(x)e^{-x^2} = \left(p_n'(x) - 2xp_n(x)\right)e^{-x^2}.$$

Let

$$p_{n+1} = p_n'(x) - 2xp_n(x).$$

Then we see that since p_n is of degree n, p_{n+1} is of degree $n+1$. Moreover

$$\frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = p_{n+1}(x)e^{-x^2}.$$

So, in fact, the Hermite polynomials satisfy:

$$H_0 = 1, \quad H_{n+1} = -\left(H_n'(x) - 2xH_n(x)\right).$$

Proposition 6. *The Hermite polynomials are orthogonal on $\mathbb{R}$ with respect to the weight function e^{-x^2}. Moreover, with respect to this weight function $||H_n||^2 = 2^n n! \sqrt{\pi}$.*

Proof: Assume $n > m \geq 0$. We compute

$$\int_{\mathbb{R}} H_n(x)H_m(x)e^{-x^2}dx = \int_{\mathbb{R}} (-1)^n \frac{d^n}{dx^n} e^{-x^2} H_m(x)dx.$$

We use integration by parts n times, noting that the rapid decay of e^{-x^2} kills all boundary terms. We therefore get

$$\int_{\mathbb{R}} e^{-x^2} \frac{d^n}{dx^n} H_m(x)dx = 0.$$

This is because the polyhomial, H_m, is of degree $m < n$. Therefore differentiating it n times results in zero. Finally, for $n = m$, we have by the same integration by parts,

$$\int_{\mathbb{R}} H_n^2(x)e^{-x^2}dx = \int_{\mathbb{R}} e^{-x^2} \frac{d^n}{dx^n} H_n(x)dx.$$

The n^{th} derivative of H_n is just the n^{th} derivative of the highest order term. By our preceding calculation, the highest order term in H_n is

$$(2x)^n.$$

Differentiating n times gives

$$2^n n!.$$

Thus

$$\int_{\mathbb{R}} H_n^2(x) e^{-x^2} dx = 2^n n! \int_{\mathbb{R}} e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

We may wish to use the following lovely fact, but we shall not prove it.

Theorem 7. *The Hermite polynomials are an orthogonal basis for $\mathcal{L}^2$ on $\mathbb{R}$ with respect to the weight function e^{-x^2}.*

2.1. **Answers to the exercises to be done oneself.**

(1) (5.2.4) Demonstrate the identity:

$$\int_0^x sJ_0(s)ds = xJ_1(x), \quad \int_0^x J_1(s)ds = 1 - J_0(x).$$

Well, one of the recurrence formulas says

$$\frac{d}{dx}(xJ_1(x)) = xJ_0(x).$$

Thus a function whose derivative is equal to $sJ_0(s)$ is the function $xJ_1(x)$. Hence we can evaluate

$$\int_0^x sJ_0(s)ds = \left. sJ_1(s)\right|_{s=0}^{s=x} = xJ_1(x).$$

Another of the recurrence formulas says

$$\frac{d}{dx}J_0(x) = -J_1(x).$$

So,

$$\int_0^x J_1(s)ds = -\left. J_0(s)\right|_{s=0}^{s=x} = J_0(0) - J_0(x) = 1 - J_0(x).$$

(2) (5.5.1) A cylinder of radius b is initially at the constant temperature A. Find the temperatures in it at subsequent times if its ends are insulated and its circular surface obeys Newton's law of cooling, $u_r + cu = 0$, $(c > 0)$. Answer:

$$u(r,t) = 2A \sum_{k \geq 1} \frac{\lambda_k J_1(\lambda_k)}{(\lambda_k^2 + b^2c^2)J_0(\lambda_k)^2} J_0\left(\frac{\lambda_k r}{b}\right) e^{-\lambda_k^2 t/b^2},$$

where λ_k is the $k^t h$ positive solution to

$$\lambda_k J_0'(\lambda_k) + bcJ_0(\lambda_k) = 0.$$

(3) (5.5.5) Solve the problem

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} = 0 \text{ in } D = \{(r,\theta,z) : 0 \leq r \leq b, 0 \leq z \leq l\}$$

$$u(r,\theta,0) = 0, \quad u(r,\theta,l) = g(r,\theta), \quad u(b,\theta,z) = 0.$$

Answer:

$$u(r,\theta,z) = \sum_{n \geq 0} \sum_{k \geq 1} (a_{kn} \cos n\theta + b_{kn} \sin n\theta) J_n\left(\frac{\lambda_{k,n} r}{b}\right) \sinh\left(\frac{\lambda_{k,n} z}{b}\right),$$

where

$$b_{k,n} = \frac{2}{b^2 \pi \sinh \lambda_{k,n}} \int_{-\pi}^{\pi} \int_0^b g(r\theta) \frac{J_n(\lambda_{k,n} r)}{J_{n+1}(\lambda_{k,n})^2} \sin n\theta r dr d\theta,$$

and similarly for $a_{k,n}$ where $\lambda_{k,n}$ is the k^{th} positive zero of J_n.

(4) (5.5.6) Find the steady-state temperature in the cylinder $0 \leq r \leq 1$, $0 \leq z \leq 1$ when the circular surface is insulated, the bottom is kept at temperature 0, and the top is kept at temperature $f(r)$. Answer:

$$u(r, z) = a_0 z + \sum_{k \geq 1} a_k J_0(\lambda_k r) \sinh(\lambda_k z),$$

where λ_k is the k^{th} positive zero of J_0,

$$a_0 = 2 \int_0^1 r f(r) dr,$$

and

$$a_k = \frac{2}{J_0(\lambda_k)^2 \sinh \lambda_k} \int_0^1 r f(r) J_0(\lambda_k r) dr, \quad k > 0.$$

(5) Eö 29 (answer is in there!)

(6) Eö 35 (answer is in there!)