FOURIER ANALYSIS & METHODS 2020.02.14

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Inhomogeneous PDE with time dependent inhomogeneity

Solve:

$$u_t - u_{xx} = tx, \quad 0 < x < 4, \quad t > 0,$$

$$\begin{aligned} u(x,0) &= v(x), \\ u_x(4,t) &= 0, \\ u(0,t) &= 0. \end{aligned}$$

Non! Sacre bleu! Tabernac![1] This is an inhomogeneous PDE *and the inhomogeneity (tx) depends on time!* A steady-state solution cannot save us. What do we do?

> **Idea:** Use a Fourier Series with non-constant coefficients to deal with time-dependent inhomogeneity.

There's a lovely way to deal with this type of inhomogeneity. We first solve the homogeneous problem.

Exercise 1. *Use separation of variables to solve the homogeneous problem:*

$$w_t - w_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$\begin{aligned} w(x,0) &= v(x) \\ w_x(4,t) &= 0 \\ w(0,t) &= 0. \end{aligned}$$

Having done this, we obtain

$$\lambda_n = -\frac{(2n+1)^2\pi^2}{64}, \quad X_n(x) = \sin(\sqrt{|\lambda_n|}x).$$

$$T_n(t) = \alpha_n e^{\lambda_n t}.$$

[1]This is how they curse in French Canada.

$$\alpha_n = \frac{\langle v, X_n \rangle}{||X_n||^2} = \frac{\int_0^4 v(x)\overline{X_n(x)}dx}{\int_0^4 |X_n(x)|^2 dx},$$

and

$$w(x,t) = \sum_{n \geq 1} T_n(t) X_n(x).$$

Now, we look for a solution to this problem:

$$\phi_t - \phi_{xx} = tx, \quad 0 < x < 4, \quad t > 0,$$

$$\phi(x,0) = 0,$$
$$\phi_x(4,t) = 0,$$
$$\phi(0,t) = 0.$$

Idea: look for a solution of the form

$$\sum_{n \geq 1} c_n(t) X_n(x).$$

So, we keep our X_n from the homogeneous problem, and we look for different $c_n(t)$ which will now be *functions of* t. We want the function to satisfy

$$u_t - u_{xx} = tx,$$

so we put the series in the left side into this PDE:

$$\sum_{n \geq 1} c_n'(t) X_n(x) - c_n(t) X_n''(x) = tx.$$

We use the fact the $X_n'' = \lambda_n X_n$, so we want to solve

$$\sum_{n \geq 1} X_n(x) \left(c_n'(t) - c_n(t)\lambda_n \right) = tx.$$

Here is where we do something clever:

Idea: write out tx as a Fourier series in terms of X_n.

The t just goes along for the ride, and

$$tx = t \sum_{n \geq 1} a_n X_n(x),$$

where

$$a_n = \frac{\langle x, X_n \rangle}{||X_n||^2} = \frac{\int_0^4 x X_n(x) dx}{\int_0^4 |X_n|^2 dx}.$$

As usual, we do not need to compute these integrals.

So, we want:

$$\sum_{n \geq 1} X_n(x) \left(c_n'(t) - c_n(t)\lambda_n \right) = tx = \sum_{n \geq 1} t X_n(x) a_n.$$

We equate the coefficients of X_n:

$$(c_n'(t) - \lambda_n c_n(t)) = t a_n.$$

This is an ODE for $c_n(t)$. We also want the IC, $c_n(0) = 0$. The solution to the homogeneous ODE,

$$f' - \lambda_n f = 0 \implies f(t) = e^{\lambda_n t} \text{ times some constant factor.}$$

A particular solution to the inhomogeneous ODE is a linear function of the form:

$$A_n t + B_n \implies A_n - \lambda_n(A_n t + B_n) = a_n t \implies A_n = \frac{-a_n}{\lambda_n}, \quad B_n = \frac{A_n}{\lambda_n} = -\frac{a_n}{\lambda_n^2}.$$

So general solutions are of the form:

$$c_n(t) = C_n e^{\lambda_n t} - \frac{a_n}{\lambda_n} t - \frac{a_n}{\lambda_n^2}, \quad \text{for some constant } C_n.$$

To obtain the initial condition that $c_n'(0) = 0$, we see that we need

$$C_n = \frac{a_n}{\lambda_n^2}.$$

Thus, we have found

$$c_n(t) = \frac{a_n}{\lambda_n^2} e^{\lambda_n t} - \frac{a_n}{\lambda_n} t - \frac{a_n}{\lambda_n^2}.$$

Therefore the solution we seek is

$$u(x,t) = \sum_{n \geq 1} c_n(t) X_n(x),$$

and the full solution to the original problem is

$$U(x,t) = w(x,t) + u(x,t).$$

2. Solving problems where the space variable is in an unbounded region

We will now develop a set of techniques which can be used for solving partial differential equations when the space variable is in an unbounded region. It is straightforward to generalize the definitions of $\mathcal{L}^1$ and $\mathcal{L}^2$ to the real line.

Definition 1 (The real one)**.** *The set*

$$\mathcal{L}^1(\mathbb{R}) = \text{ the set of equivalence classes, of functions which satisfy:}$$

$$f \text{ is measurable, and } \int_{\mathbb{R}} |f(x)| dx < \infty.$$

The function g belongs to the same equivalence class as f if $g = f$ almost everywhere on $\mathbb{R}$ with respect to the Lebesgue measure on $\mathbb{R}$.

Definition 2 (The real one)**.** *The set*

$$\mathcal{L}^2(\mathbb{R}) = \text{ the set of equivalence classes of functions which satisfy:}$$

$$f \text{ is measurable, and } \int_{\mathbb{R}} |f(x)|^2 dx < \infty.$$

The function g belongs to the same equivalence class as f if $g = f$ almost everywhere on $\mathbb{R}$ with respect to the Lebesgue measure on $\mathbb{R}$.

Definition 3 (The workable definition of $\mathcal{L}^1(\mathbb{R})$)**.** *It will suffice for the purposes of this humble course to treat $\mathcal{L}^1(\mathbb{R})$ as the set of functions on $\mathbb{R}$ which satisfy*

$$\int_{\mathbb{R}} |f(x)| dx < \infty.$$

The $\mathcal{L}^1(\mathbb{R})$ norm is then defined to be

$$||f||_{\mathcal{L}^1} = \int_{\mathbb{R}} |f(x)| dx.$$

The set of such functions, denoted by $\mathcal{L}^1(\mathbb{R})$, is a complete normed vector space but not *a Hilbert space. A complete normed vector space is also known as a Banach space.*

Definition 4 (The workable definition of $\mathcal{L}^2(\mathbb{R})$)**.** *It will suffice for the purposes of this humble course to treat $\mathcal{L}^2(\mathbb{R})$ as the set of functions on $\mathbb{R}$ which satisfy*

$$\int_{\mathbb{R}} |f(x)|^2 dx < \infty.$$

This set of functions, denoted by $\mathcal{L}^2(\mathbb{R})$, is a Hilbert space with the scalar product:

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} d\mu.$$

Hence, by definition, the norm on $\mathcal{L}^2(\mathbb{R})$ is

$$||f||_{\mathcal{L}^2(\mathbb{R})} = \sqrt{\int_{\mathbb{R}} |f(x)|^2 dx}.$$

A lot of things which are true for $\mathcal{L}^2$ on a finite interval are no longer true on $\mathcal{L}^2(\mathbb{R})$. For example, the functions

$$e^{inx}, \sin(x), \cos(x)$$

are all neither in $\mathcal{L}^1(\mathbb{R})$ nor in $\mathcal{L}^2(\mathbb{R})$. Furthermore, there is no relationship between $\mathcal{L}^1(\mathbb{R})$ and $\mathcal{L}^2(\mathbb{R})$. There are functions which are in $\mathcal{L}^1(\mathbb{R})$ but not in $\mathcal{L}^2(\mathbb{R})$:

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \sqrt{x} & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

is in $\mathcal{L}^1(\mathbb{R})$ but it is *not* in $\mathcal{L}^2(\mathbb{R})$.

Exercise 2. *Verify that this function is in $\mathcal{L}^1(\mathbb{R})$ but not in $\mathcal{L}^2(\mathbb{R})$. Compute its $\mathcal{L}^1(\mathbb{R})$ norm.*

On the other hand, the function

$$f(x) = \begin{cases} 0 & x \leq 1 \\ \frac{1}{x} & x > 1 \end{cases}$$

is in $\mathcal{L}^2(\mathbb{R})$ but not in $\mathcal{L}^1(\mathbb{R})$.

Exercise 3. *Verify that this function is in $\mathcal{L}^2(\mathbb{R})$ but not in $\mathcal{L}^1(\mathbb{R})$. Compute its $\mathcal{L}^2(\mathbb{R})$ norm.*

The function

$$e^{-|x|}$$

is in both $\mathcal{L}^1(\mathbb{R})$ and in $\mathcal{L}^2(\mathbb{R})$.

Exercise 4. *Verify that this function is in both $\mathcal{L}^1(\mathbb{R})$ and $\mathcal{L}^2(\mathbb{R})$. Compute its $\mathcal{L}^1$ and $\mathcal{L}^2$ norms. Come up with your own examples of functions which are*

(1) In $\mathcal{L}^1(\mathbb{R})$ but not in $\mathcal{L}^2(\mathbb{R})$.
(2) In $\mathcal{L}^2(\mathbb{R})$ but not in $\mathcal{L}^1(\mathbb{R})$.
(3) In both $\mathcal{L}^1(\mathbb{R})$ and $\mathcal{L}^2(\mathbb{R})$.

So, all we can say is that

$$\mathcal{L}^1(\mathbb{R}) \not\subset \mathcal{L}^2(\mathbb{R}), \quad \mathcal{L}^2(\mathbb{R}) \not\subset \mathcal{L}^1(\mathbb{R}), \quad \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R}) \neq \emptyset.$$

So, we're in a whole new territory here. To begin we shall define the convolution. This will be super important for solving the heat equation on the real line.

Definition 5. *The convolution of f and g is a function $f * g : \mathbb{R} \to \mathbb{C}$ defined by*

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy,$$

whenever the integral on the right exist.

Proposition 6. *Assume that f and g are both in $\mathcal{L}^2(\mathbb{R})$. Then*

(1) $|f * g(x)| \leq ||f||||g||$ *for all* $x \in \mathbb{R}$
(2) $f * (ag + bh) = af * g + bf * h$ *for all* $a, b \in \mathbb{C}$
(3) $f * g = g * f$
(4) $f * (g * h) = (f * g) * h$

Proof: This is useful to do because it helps to familiarize oneself with the convolution. We first estimate

$$|f * g(x)| = \left| \int_{\mathbb{R}} f(x-y)g(y)dy \right| \leq \int_{\mathbb{R}} |f(x-y)||g(y)|dy.$$

The point $x \in \mathbb{R}$ is fixed and arbitrary, so we define a function

$$\phi(y) = f(x-y).$$

Then

$$|f * g(x)| \leq \int_{\mathbb{R}} |\phi(y)||g(y)|dy \leq ||\phi||||g||.$$

We compute

$$||\phi||^2 = \int_{\mathbb{R}} |f(x-y)|^2 dy = -\int_{\infty}^{-\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |f(t)|^2 dt = ||f||^2.$$

Above, we used the substitution $t = x - y$ so $dt = -dy$, and the integral got reversed. The $-$ goes away when we re-reverse the integral. So, in the end we see that

$$|f * g(x)| \leq ||f||||g||$$

as desired. The second property follows simply by the linearity of the integral itself. For the third property, we will use substitution again:

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

We want to get $g(x-z)$ so we define

$$y = x - z \implies x - y = z, \quad dz = -dy.$$

Hence,

$$f * g(x) = -\int_{\infty}^{-\infty} f(z)g(x-z)dz = \int_{-\infty}^{\infty} g(x-z)f(z)dz = g * f(x).$$

We do something rather similar in the fourth property:

$$f * (g * h)(x) = \int_{\mathbb{R}} f(x-y) \int_{\mathbb{R}} g(y-z)h(z)dzdy.$$

For the other term we have

$$(f*g)*h(x) = \int_{\mathbb{R}} (f*g)(x-y)h(y)dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y-z)g(z)h(y)dzdy.$$

So, we define

$$t = y - z \implies x - y = x - t - z, \quad dt = dy.$$

Then

$$f*(g*h)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t-z)g(t)h(z)dzdt.$$

Finally, we call $z = y$ and $t = z$ (sorry if this gives you a headache!) because they are just names, and then we get

$$f*(g*h)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y-z)g(y)h(z)dzdy.$$

If you're worried about the order of integration, don't be. Since everything is in $\mathcal{L}^2$, these integrals converge absolutely, so those Italian magicians, Fubini & Tonelli allow us to do the switch-a-roo with the integrals as much as we like.

One of the useful features of convolution is that we can use it to smooth out non-smooth functions. This is known as *mollification,* which comes from the verb, to mollify, which means to make smooth.[2]

Proposition 7 (Mollification)**.** *If $f \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$, $f' \in \mathcal{L}^2(\mathbb{R})$, and $g \in \mathcal{L}^2(\mathbb{R})$, then $f*g \in \mathcal{C}^1(\mathbb{R})$. Moreover $(f*g)' = f'*g$.*

Proof: Everything converges beautifully so just stick that differentiation right under the integral defining

$$f*g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Hence

$$(f*g)'(x) = \int_{\mathbb{R}} f'(x-y)g(y)dy = f'*g(x).$$

If you are not satisfied with this explanation, a rigorous proof can be obtained using the Dominated Convergence Theorem, but that is a theorem which we cannot prove in the context of this humble course.

2.0.1. *An example.* Let's compute a convolution. Let $f(x) = \frac{1}{1+x^2}$ and

$$g(x) = \begin{cases} 1 & |x| < 3 \\ 0 & |x| > 3 \end{cases}.$$

The function g is not differentiable at the points ± 3. The function f is perfectly smooth on $\mathbb{R}$. Let's convolve them!

$$f*g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy = \int_{\mathbb{R}} \frac{1}{1+(x-y)^2} g(y)dy = \int_{-3}^{3} \frac{1}{1+(x-y)^2} dy.$$

[2] One can mollify garlic, tahini, chickpeas, soy sauce, olive oil, oregano, black pepper, lemon juice, in suitable proportions, together with a bit of hot sauce like Cholula, Tabasco, or Sriracha, to make hummus.

If we dig deep into our calculus memory, we vaguely recall that

$$(\arctan(t))' = \frac{1}{1+t^2}.$$

So, this integral becomes:

$$-\arctan(x-y)\Big|_{-3}^{3} = -\arctan(x-3) + \arctan(x+3).$$

This is indeed a smooth function of x.

3. The Fourier transform

One of the reasons that the convolution is so nice is because it plays well with the Fourier transform. So let us define this Fourier transform.

Proposition 8. *Assume that* $f \in \mathcal{L}^1(\mathbb{R})$*. Then*

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$$

is a well-defined complex number for any $\xi \in \mathbb{R}$*.*

Proof: Simply estimate

$$\left| \int_{\mathbb{R}} e^{-ix\xi} f(x) dx \right| \le \int_{\mathbb{R}} |f(x)| dx < \infty.$$

3.1. **Example of computing a Fourier transform.**

Let us get a feel for this by computing a Fourier transform. Consider the function $f(x) = e^{-a|x|}$ where $a > 0$. Then it is certainly in $\mathcal{L}^1(\mathbb{R})$ so we ought to be able to compute its Fourier transform. This is by definition

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} e^{-a|x|} dx = \int_{-\infty}^{0} e^{-ix\xi} e^{ax} dx + \int_{0}^{\infty} e^{-ix\xi} e^{-ax} dx.$$

We compute these integrals by finding a primitive for the integrand:

$$\hat{f}(\xi) = \left. \frac{e^{x(a-i\xi)}}{a-i\xi} \right|_{-\infty}^{0} + \left. \frac{e^{x(-a-i\xi)}}{-a-i\xi} \right|_{0}^{\infty}$$

$$= \frac{1}{a-i\xi} + \frac{1}{a+i\xi} = \frac{a+i\xi+a-i\xi}{a^2+\xi^2} = \frac{2a}{a^2+\xi^2}.$$

3.2. **Answers for this week's exercises to be done oneself.**

(1) (Eo 25, 27, 3) Please see the end of the Eö document! It has answers!
(2) (4.4:1)

$$u(x,y) = \frac{8l^2}{\pi^3} \sum_{n \ge 1} \frac{1}{(2n-1)^3 \sinh((2n-1)\pi)} \sin\left(\frac{(2n-1)\pi x}{l}\right) \sinh\left(\frac{(2n-1)\pi y}{l}\right).$$

(3) (4.2.2) Here we define first:

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{(2n-1)\pi x}{2l}\right) dx.$$

Then the answer to this one is:

$$u(x,t) = C + \sum_{n\geq 1} \left(b_n - \frac{4C}{\pi(2n-1)} \right) \exp\left(-\frac{(n-1/2)^2\pi^2 kt}{l^2} \right) \sin\left(\left(n - \frac{1}{2} \right) \frac{\pi x}{l} \right).$$

(4) (4.3.1) And the answer is... Geez Folland where is the answer? Oh, right this one is to "verify" etc. Well, the way I find this easiest to do is to re-write using the angle addition formula for the sine:

$$b_n(t) = \frac{l}{n\pi c} \int_0^t \sin(n\pi ct/l)\cos(-n\pi cs/l)\beta_n(s)ds + \frac{l}{n\pi c} \int_0^t \cos(n\pi ct/l)\sin(-n\pi cs/l)\beta_n(s)ds.$$

Then we can take out the s-independent terms to the front of the integral, so that

$$b_n(t) = \sin(n\pi ct/l)\frac{l}{n\pi c} \int_0^t \cos(-n\pi cs/l)\beta_n(s)ds + \cos(n\pi ct/l)\frac{l}{n\pi c} \int_0^t \sin(-n\pi cs/l)\beta_n(s)ds.$$

Now we can compute the derivatives and verify the formulas using the product rule together with the fundamental theorem of calculus. Please just ask if you have questions about how this works. Also, if you solved in a different way but ended up correct, that's just peachy too!

(5) (4.4.7) Wow, this answer is long. Let

$$g(r) = \sum c_n \sin\left(\frac{n\pi \log r}{\log r_0} \right)$$

and

$$h(r) = \sum d_n \sin\left(\frac{n\pi \log r}{\log r_0} \right),$$

then

$$u(r,\theta) = \sum_{n\geq 1} (a_n e^{n\pi\theta/\log r_0} + b_n e^{-n\pi\theta/\log r_0}) \sin\left(\frac{n\pi \log r}{\log r_0} \right),$$

where

$$a_n + b_n = c_n, \quad a_n e^{n\pi\beta/\log r_0} + b_n e^{-n\pi\beta/\log r_0} = d_n.$$

Happy Weekend! ♡