1. Show that the up- and down-sampling operators (↑ 2) and (↓ 2) are linear but not time-invariant.

2. Verify the following Fourier transformation

\[ \Pi(x) \text{ sign } x \supset -i \sin \frac{\pi s}{2} \text{ sinc } \left( \frac{s}{2} \right). \]

3. Verify that \( f \in S \), if \( f \) is given by

\[ f(x) = \begin{cases} 0, & x \notin (a,b) \\ e^{-\frac{1}{(x-a)^2}} - e^{-\frac{1}{(x-b)^2}}, & x \in (a,b). \end{cases} \]

4. Verify that, for \( j \neq 0 \), each function \( f_{j+1} \in V_{j+1} \) can be written as \( f_{j+1} = f_j + d_j \) where \( d_j \in W_j \). (For \( j = 0 \) the statement is true by definition.)

5. Let \( \varphi \) be a scaling function in some (ON) MRA which vanishes outside \((0, L)\) and this interval cannot be taken smaller. Show that the corresponding low-pass filter has length \( L + 1 \).
void!
1. Solution:

\((\downarrow 2)x = (\ldots, x_{-4}, x_{-2}, 0, x_2, x_4, \ldots)\)

Thus \(y = (\downarrow 2)x\) is given by \(y_n = x_{2n}, n \in \mathbb{Z}\). Combining with the definition

\[(Dx)_n = x_{n-1}, \quad \text{yields} \quad \left((\downarrow 2)(Dx)\right)_n = (Dx)_{2n} = x_{2n-1},\]

whereas

\[(D(\downarrow 2)x)_n = \left((\downarrow 2)x\right)_{n-1} = x_{2n-2}.\]

Thus the operator \((\downarrow 2)\) is not time invariant. However,

\[
\begin{cases}
(\downarrow 2)(x+y)_n = (x+y)_{2n} = (x)_{2n} + (y)_{2n} = \left((\downarrow 2)x\right)_n + \left((\downarrow 2)y\right)_n \\
(\downarrow 2)(ax)_n = (ax)_{2n} = ax_{2n} = a\left((\downarrow 2)x\right)_n 
\end{cases}
\]

i.e. \((\downarrow 2)\) is linear. Reverse arrow would give the claim for up-sampling.

2. Solution: Using the definitions of \(\text{sgn}, \text{sinc}\), and Fourier transform, we may write

\[
\Pi(x) \text{ sign } x = \Pi \left(2(x - \frac{1}{4})\right) - \Pi \left(2(x + \frac{1}{4})\right) \supset e^{-2\pi i \frac{1}{4}} \frac{1}{2} \hat{\Pi}(\frac{s}{2}) - e^{2\pi i \frac{1}{4}} \frac{1}{2} \hat{\Pi}(\frac{s}{2})
\]

\[
= \frac{1}{2} \left(e^{-i \frac{s}{2} x} - e^{i \frac{s}{2} x} \right) \frac{1}{2} \hat{\Pi}(\frac{s}{2}) = -i \sin \frac{\pi s}{2} \text{sinc} \left(\frac{s}{2}\right).
\]

3. Solution: Let

\[
\psi(x) = \begin{cases}
0, & x \leq 0 \\
e^{-\frac{1}{x^2}}, & x > 0
\end{cases}
\]

The function \(\psi\) is infinitely many times differential for both \(x < 0\) and \(x > 0\). Further

\[
\lim_{x \to 0^+} e^{-1/x^2} = 0.
\]

Hence \(\psi\) is continuous at the origin. Differentiating we get

\[
\psi'(x) = \frac{2}{x^3} e^{-1/x^2} \quad \text{for} \quad x > 0 \quad \text{and} \quad \lim_{x \to 0^+} \psi'(x) = 0.
\]

Consequently \(\psi'(0) = 0\) and \(\psi \in C^1\). Using induction procedure we can get that \(\psi \in C^n\) and \(\psi^{(n)}(0) = 0\) for all \(n\). Thus

\[
\varphi(x) = \psi(x - a)\psi(x - b) \in C^\infty,
\]

and

\[
\varphi(x) = 0 \quad \text{for} \quad x \leq a \quad \text{and} \quad x \geq b, \quad \varphi(x) > 0 \quad \text{for} \quad a < x < b, \quad \text{i.e.} \quad \varphi \in \mathcal{S}.
\]
4. Solution:
By definitions, $V_j$ is generated by $\{\varphi(2^jt-k)\}_{k \in \mathbb{Z}}$ and $W_j$ is generated by $\{\psi(2^jt-k)\}_{k \in \mathbb{Z}}$. Moreover $f_1 \in V_1$ can be written as $f_1 = f_0 + d_0$ with $f_0 \in V_0$, $d_0 \in W_0$. Let $f_{j+1} \in V_{j+1}$, then $f_{j+1}$ can be written as

$$f_{j+1}(t) = \sum_k C_{j+1,k} \varphi(2^{j+1}t - k).$$

Thus

$$f_{j+1}(2^{-j}t) = \sum_k C_{j+1,k} \varphi(2t - k) \in V_1,$$

which yields

$$f_{j+1}(2^{-j}t) = \sum_k C_{0,k} \varphi(t - k) + \sum_k w_{0,k} \psi(t - k)$$

and hence

$$f_{j+1}(t) = \sum_k C_{0,k} \varphi(2^jt - k) + \sum_k w_{0,k} \psi(2^jt - k) = f_j(t) + d_j(t),$$

where $f_j \in V_j$ and $d_j \in W_j$.

5. Solution: Since $\varphi(t) = 2 \sum_k h_k \varphi(2t - k)$, we get by orthogonality

$$h_k = \int_{-\infty}^{\infty} \varphi(t) \varphi(2t - k) dt, \quad \varphi(2t - k) = 0, \quad \text{outside } \left[\frac{k}{2}, \frac{L+k}{2}\right].$$

Thus

$$h_k = 0, \quad \text{if } \frac{k}{2} \geq L, \quad \text{and } \frac{L+k}{2} \leq 0, \quad \text{i.e., if } k \leq -L \text{ or } k \geq 2L.$$

Hence

$$\varphi(t) = 2 \sum_{k=-L+1}^{2L-1} h_k \varphi(2t - k).$$

This yields on $\left(-\frac{L+1}{2}, -\frac{L+2}{2}\right)$ we have

$$\varphi(t) = 2h_{-L+1} \varphi(2t + L - 1), \quad \text{so that } h_{-L+1} = 0, \quad \text{if } L < 1.$$ 

Note that $h_k = 0$ if $\frac{k}{2} < 0$, $(k < 0)$ and if $\frac{k+L}{2} > L$ $(k > L)$.

On the other hand we must have $h_0 \neq 0$ and $h_L \neq 0$. Since $\varphi(t)$ is not identically 0 in $(0, 1/2)$ and $(L - 1/2, L)$. Consequently, the filter $H$ has length $L + 1$. 

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