# FOURIER ANALYSIS & METHODS

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Let's look at another example. Consider a circular shaped rod, like a rod that's been bent into a circle. Let's mathematicize it! To specify points on the rod, we just need to know the angle at the point. For this reason, we use the real variable $x$ for the position, where $x$ gives us the angle at the point on the rod. We use the variable $t \geq 0$ for time. The function $u(x,t)$ is the temperature on the rod at position $x$ at time $t$.

The heat equation (homogeneous, which means no sources or sinks) tells us that:

$$u_t = k u_{xx},$$

for some constant $k > 0$. At this point our only techniques are separation of variables and superposition. We first use separation of variables to find solutions. So, let us do the same first step as we did in solving the homogeneous wave equation. It's just a means to an ends, by writing

$$u(x,t) = X(x)T(t).$$

Plug it into the heat equation:

$$T'(t)X(x) = kX''(x)T(t).$$

We want to *separate variables,* so we want all the $t$-dependent bits on the left say, and all the $x$-dependent bits on the right. This can be achieved by dividing both sides by $X(x)T(t)$,

$$\frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)}.$$

We now know that both sides must be constant. Let us call the constant $\lambda$, so that

$$\frac{T'}{T} = \lambda = k\frac{X''}{X}.$$

**Exercise 1.** *In your own words, explain why both sides of the equation must be constant.*

Now, we need to pick a side to begin... We actually have some information which is hiding inside the *geometry* of the problem. The geometry is referring to the $x$ variable. What can you say about the angle $x$ on the rod and the angle $x + 2\pi$ on the rod? They are the same. This means that our temperature function must be the same at $x$ and at $x+2\pi$. So, we must have

$$X(x+2\pi) = X(x).$$

We can repeat this, obtaining

$$X(x+2\pi n) = X(x) \quad \forall n \in \mathbb{Z}.$$

This means that $X$ is a periodic function with period equal to $2\pi$. So, we have a bit of extra information about it. The equation for $X$ is:

$$X''(x) = \frac{\lambda}{k} X(x)$$

for a constant $\lambda$.

**Exercise 2.** ***Case 1:*** *Show that if $\lambda = 0$, there is no solution to $X''(x) = 0$ which is $2\pi$ periodic, other than the constant solutions.*

**Case 2:** If $\lambda > 0$, then a basis of solutions is,

$$\{e^{\sqrt{\lambda}x/\sqrt{k}}, e^{-\sqrt{\lambda}x/\sqrt{k}}\}.$$

So, we can write

$$X(x) = ae^{\sqrt{\lambda}x/\sqrt{k}} + be^{-\sqrt{\lambda}x/\sqrt{k}}.$$

For the $2\pi$ periodicity to hold, we need

$$X(0) = X(2\pi) \implies a+b = ae^{\sqrt{\lambda}2\pi/\sqrt{k}} + be^{-\sqrt{\lambda}2\pi/\sqrt{k}} \implies a(e^{\sqrt{\lambda}2\pi/\sqrt{k}} - 1) = b(1 - e^{-\sqrt{\lambda}2\pi/\sqrt{k}})$$

$$\implies a = b\frac{(1 - e^{-\sqrt{\lambda}2\pi/\sqrt{k}})}{e^{\sqrt{\lambda}2\pi/\sqrt{k}} - 1}.$$

We also need

$$X(-2\pi) = X(0) \implies a+b = ae^{-\sqrt{\lambda}2\pi/\sqrt{k}} + be^{\sqrt{\lambda}2\pi/\sqrt{k}} \implies a(e^{-\sqrt{\lambda}2\pi/\sqrt{k}} - 1) = b(1 - e^{\sqrt{\lambda}2\pi/\sqrt{k}})$$

$$\implies a = b\frac{1 - e^{\sqrt{\lambda}2\pi/\sqrt{k}}}{e^{-\sqrt{\lambda}2\pi/\sqrt{k}} - 1}.$$

So, we have two equations for $a$, therefore they should be equal:

$$a = b\frac{1 - e^{-\sqrt{\lambda}2\pi/\sqrt{k}}}{e^{\sqrt{\lambda}2\pi/\sqrt{k}} - 1} = b\frac{1 - e^{\sqrt{\lambda}2\pi/\sqrt{k}}}{e^{-\sqrt{\lambda}2\pi/\sqrt{k}} - 1}.$$

If $b = 0$ then $a = 0$ so the whole solution is the zero solution. If $b \neq 0$ then we must have

$$\frac{1 - e^{-\sqrt{\lambda}2\pi/\sqrt{k}}}{e^{\sqrt{\lambda}2\pi/\sqrt{k}} - 1} = \frac{1 - e^{\sqrt{\lambda}2\pi/\sqrt{k}}}{e^{-\sqrt{\lambda}2\pi/\sqrt{k}} - 1}.$$

Changing the sign of the top and bottom on the right side, this is equivalent to:

$$\frac{1 - e^{-\sqrt{\lambda}2\pi/\sqrt{k}}}{e^{\sqrt{\lambda}2\pi/\sqrt{k}} - 1} = \frac{e^{\sqrt{\lambda}2\pi/\sqrt{k}} - 1}{1 - e^{-\sqrt{\lambda}2\pi/\sqrt{k}}}.$$

Call the left side $\star$. Then the right side is $\frac{1}{\star}$. So the equation is

$$\star = \frac{1}{\star} \implies \star^2 = 1 \implies \star = \pm 1.$$

**Exercise 3.** *Show that $\star > 0$.*

If

$$\star = 1 \implies 1 - e^{-\sqrt{\lambda}2\pi/\sqrt{k}} = e^{\sqrt{\lambda}2\pi/\sqrt{k}} - 1 \implies 2 = e^{\sqrt{\lambda}2\pi/\sqrt{k}} + e^{-\sqrt{\lambda}2\pi/\sqrt{k}}.$$

I don't like the negative exponent thing (it is really a fraction), so I am going to multiply by $e^{\sqrt{\lambda}2\pi/\sqrt{k}}$. Also, doing this turns it into a quadratic equation:

$$2e^{\sqrt{\lambda}2\pi/\sqrt{k}} = e^{4\pi\sqrt{\lambda}/\sqrt{k}} + 1 \iff e^{4\pi\sqrt{\lambda}/\sqrt{k}} - 2e^{2\pi\sqrt{\lambda}/\sqrt{k}} + 1 = 0$$

Now we can factor this equation because the left side is

$$(e^{2\pi\sqrt{\lambda}/\sqrt{k}} - 1)^2 = 0 \implies e^{2\pi\sqrt{\lambda}/\sqrt{k}} = 1 \iff 2\pi\sqrt{\lambda}/\sqrt{k} = 0 \lightning.$$

That $\lightning$ indicates a contradiction. Therefore, in the case where $\lambda > 0$, the only solution which is $2\pi$ periodic is the zero solution.

Hence, we are left with **Case 3:** $\lambda < 0$. Then, a basis of solutions is

$$\{\sin(\sqrt{|\lambda|}x/\sqrt{k}), \cos(\sqrt{|\lambda|}x/\sqrt{k}).$$

We need these solutions to be $2\pi$ periodic. They will be as long as $\sqrt{|\lambda|}/\sqrt{k}$ is an integer. So we need

$$\lambda < 0, \quad \frac{\sqrt{|\lambda|}}{\sqrt{k}} = n \in \mathbb{Z} \implies \lambda_n = -n^2 k.$$

Hence, our solution

$$X_n(x) = a_n \cos(nx) + b_n \sin(nx), \quad n \in \mathbb{N}_0.$$

**Exercise 4.** *Show that allowing complex coefficients, it is equivalent to use a basis of solutions*

$$\{e^{\pi inx}\}_{n\in\mathbb{Z}}.$$

*Find $A_n$ and $B_n$ in terms of $a_n$ and $b_n$ so that*

$$X_n(x) = A_n e^{inx} + B_n e^{-inx}.$$

Now, we can solve for the partner function, $T_n(t)$. Since

$$\frac{T_n'(t)}{T_n(t)} = \lambda_n = -n^2 k,$$

the equation for $T_n$ is

$$T_n'(t) = -n^2 k T_n(t).$$

Consequently,

$$T_n(t) = e^{-n^2 kt} \text{ up to constant factor.}$$

So, we now have found the solutions

$$u_n(x,t) = X_n(x)T_n(t) = e^{-n^2 kt}(a_n \cos(nx) + b_n \sin(nx)).$$

These solutions satisfy the heat equation

$$\partial_t u_n - k\partial_{xx} u_n = 0.$$

Let us define the heat operator for functions of one real variable and one time variable,

$$\Xi := \partial_t - k\partial_{xx}.$$

Then we have

$$\Xi u_n(t) = 0 \forall n \in \mathbb{N}_0.$$

Consequently, we can use the superposition principle to smash all these solutions we have found into a super solution

$$u(x,t) = \sum_{n\geq 0} u_n(x,t) = \sum_{n\geq 0} e^{-n^2 tk}(a_n \cos(nx) + b_n \sin(nx)).$$

We do this because we do not know how many of the $u_n$ functions we will need. In case we don't end up needing them all, then their coefficients will be zero, so they will just disappear on their own anyways. Let's think about the physics. The rod has some temperature function at time $t = 0$, which we call $u_0(x)$. Then $u_0(x)$ is also a $2\pi$ periodic function. We would like

$$u(x,0) = u_0(x) \iff \sum_{n\geq 0} a_n \cos(nx) + b_n \sin(nx) = u_0(x).$$

So, given $u_0(x)$, can we find $a_n$ and $b_n$ so that this is true?

Fourier made the bold statement that we can do this. It took a long time to rigorously prove him right (like 100 years, because this whole theory about Hilbert spaces, measure theory, and functional analysis needed to get developed by Hilbert & his contemporaries).

1.1. **Introduction to Fourier Series of periodic functions.** If we have a finite one dimensional, connected set, then we can always mathematicize it as either (1) a bounded interval or (2) a circle. When we take a bounded interval of length $2\ell$, and we take any function whatsoever on that interval, we can always extend it to the rest of $\mathbb{R}$ to be $2\ell$ periodic, by simply repeating its values from the interval. Hence, for both of these contexts we can do everything we desire with periodic functions.

**Definition 1.** A function $f : \mathbb{R} \to \mathbb{R}$ is periodic with period $p$ iff for all $x \in \mathbb{R}$, $f(x+p) = f(x)$, and moreover, $p > 0$ is the smallest real number for which this is true.

For example, $\sin(x)$ is periodic with period $2\pi$. Our heat equation examples, $f_n(x) = a_n \cos(nx) + b_n \sin(nx)$ are periodic with period $2\pi/n$. We shall prove a super useful little lemma about periodic functions and their integrals.

**Lemma 2** (Integration of periodic functions lemma)**.** *If $f$ is periodic with period $p$ then for any $a \in \mathbb{R}$*

$$\int_a^{a+p} f(x)dx$$

*is the same.*

**Exercise 5.** *Give an example for how this fails to be true if the function $f$ is* not *periodic. That is, take some non-periodic function and show that integrating it from say $a$ to $a+p$ is not the same as integrating it from $c$ to $c+p$.*

**Proof:** If we think about it, we want to show that the function

$$g(a) := \int_a^{a+p} f(x)dx$$

is a constant function. This looks awfully similar to the fundamental theorem of calculus. Now, this statement above is not true for non-periodic functions. So, we're going to need to use the assumption that $f$ is periodic with period $p$. This tells us that $f$ has the same value at both endpoints of the integral, so

$$f(a) = f(a+p) \implies f(a+p) - f(a) = 0.$$

Now, since we want to consider $a$ as a variable, we don't want it at both the top and the bottom of the integral defining $g$. Instead, we can use linearity of integration to write

$$g(a) = \int_0^{a+p} f(x)dx - \int_0^a f(x)dx.$$

Then, using the fundamental theorem of calculus on each of the two terms on the right,

$$g'(a) = f(a+p) - f(a) = 0.$$

Above, we use the fact that $f$ is periodic with period $p$. Hence, $g'(a) \equiv 0$ for all $a \in \mathbb{R}$. This tells us that $g$ is a constant function, so its value is the same for all $a \in \mathbb{R}$.

So you survived a bit of theory, now let's return to our physical motivation! We wanted to find coefficients so that the $u(x,t)$ we found to solve the heat equation would match up with the initial data, $u_0(x)$. If it does, then (using some advanced PDE theory beyond the scope of this humble course), $u(x,t)$ is indeed THE UNIQUE solution to the heat equation with initial data $u_0(x)$. Hence, $u(x,t)$ actually tells us the temperature on the rod at position $x$ at time $t$. Cool. So, setting $t = 0$ in the definition of $u(x,t)$ we want

$$u_0(x) = \sum_{n \geq 0} a_n \cos(nx) + b_n \sin(nx). \tag{1.1}$$

It is totally equivalent to work with complex exponentials, because

$$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}.$$

**Exercise 6.** *Show that we can write $u_0(x)$ as a series above in (1.1) if and only if we can write*

$$u_0(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

*Moreover, show that*

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{1}{2}(a_n - ib_n), \quad n \geq 1, \quad c_n = \frac{1}{2}(a_n + ib_n), n \leq -1.$$

*Finally, use this to show that*

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n},\, n \geq 0, \quad b_n = i(c_n - c_{-n}),\, n \geq 0.$$

It is slightly more convenient for these purposes to do the calculation using the $\{e^{inx}\}_{n \in \mathbb{Z}}$ basis. This will be elucidated in a moment. The equation we want to obtain is:

$$u_0(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

The object on the right is a sum of coefficients $c_n \in \mathbb{C}$ times functions $e^{inx}$. It is simply a linear combination of the functions $e^{inx}$. If we could show that in a suitable sense these functions for a sort of "basis" then we should be able to expand our function $u_0$ in terms of this basis. Sure, the basis is infinite, so, you've graduated to "linear algebra for adults," in which your vectors are now infinite dimensional. [1] To continue with the linear algebra concept, we need a notion of dot product, in order to expand $u_0$ in terms of our basis functions $e^{inx}$. This is obtained using something called a scalar product, or dot product, or inner product: they all mean the same thing.

**Definition 3.** For two functions, $f$ and $g$, which are real or complex valued functions defined on $[a,b] \subset \mathbb{R}$, we define their scalar product to be

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx.$$

We say that $f$ and $g$ are orthogonal if $\langle f, g \rangle = 0$. We define the $L^2([a,b])$ *norm* of a function to be

$$||f||_{L^2([a,b])} = \sqrt{\langle f, f \rangle}.$$

OBS! Learn this definition right now!!!! It is really important. Every detail:

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx, \quad ||f||^2 = \langle f, f \rangle.$$

Now, if you wonder *why* it is defined this way, that is because defining things this way has the very pleasant consequence that it *works.* Meaning, when we define things this way, we are able to use the separation of variables technique to solve the PDEs.

## 2. Exercises to be done by oneself: Hints

1.1.1 Show that $u(x,t) = t^{-1/2}e^{-x^2/(4kt)}$ satisfies the heat equation

$$u_t = ku_{xx}.$$

**Hint:** Use the product rule when you're differentiating with respect to $t$. When you're differentiating with respect to $x$, remember that from $x$'s perspective, $t$ is just a constant.

1.2.5(a) Show that for $n = 1, 2, 3, \ldots$ $u_n(x,y) = \sin(n\pi x)\sinh(n\pi y)$ satisfies

$$u_{xx} + u_{yy} = 0, \quad u(0,y) = u(1,y) = u(x,0) = 0.$$

**Hint:** Use the product rule and remember that in the eyes of $x$, $\sinh(n\pi y)$ is constant. Similarly, in the eyes of $y$, $\sin(n\pi x)$ is constant.

1.3.5 By separation of variables, derive the solutions $u_n(x,y) = \sin(n\pi x)\sinh(n\pi y)$ of

$$u_{xx} + u_{yy} = 0, \quad u(0,y) = u(1,y) = u(x,0) = 0.$$

**Hint:** Start by writing $u(x,y) = X(x)Y(y)$. Plop it into the PDE. Get all the $x$ dependent terms to one side of the equation and the $y$ dependent terms to the other side. (probably do this by dividing by $XY$). Solve for $X$ first. Use the conditions on $X(0) = X(1) = 0$. (Why?) Then once you have found your $X$s (there will be many!) find their partner functions. Use the condition $Y(0) = 0$ (Why?) to help with this.

[1] Grigori Rozenblioum, who taught this class for many years, and is in general an awesome mathematician, used to say "If you can pass this course, then you've earned the right to buy Vodka at Systembolaget, regardless of your actual age."

1.3.7 Use separation of variables to find an infinite family of independent solutions to

$$u_t = ku_{xx}, \quad u(0,t) = 0, \quad u_x(\ell, t) = 0,$$

representing heat flow in a rod with one end held at temperature zero and the other end insulated. **Hint:** Start by writing $u(x,t) = X(x)T(t)$. Follow the same type of procedure as for the preceding problem, but now you have the conditions on $X$ that $X(0) = 0$, $X'(\ell) = 0$ (Why?) Find the $X$ first (there will be many!), and then use these to find their partner functions. It will be kind of similar to the example from lecture today, but the boundary conditions are different, so this will change things.

## References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).