Chalmers/GU
Mathematics

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date:	17–01–10
Time:	8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Michael Patriksson, tel. 709-581812
Result announced:	17–01–30
	Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned}
\text{maximize} \quad & z = 5x_1 + 4x_2, \\
\text{subject to} \quad & x_1 \leq 7, \\
& x_1 - x_2 \leq 8. \\
& x_1, \quad x_2 \geq 0.
\end{aligned}$$

(2p) a) Solve the problem using phase I and phase II of the simplex method. If the problem has an optimal solution, then present the optimal solution in both the original variables and in the variables used in the standard form. If the problem is unbounded, then use your calculations to find a direction of unboundness in both the original variables and in the variables in the standard form.
Aid: Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) Add a constraint to the linear program considered to obtain a uniquely solvable linear program. Present the optimal solution.

(3p) Question 2

(finiteness of the simplex algorithm)

Establish the following statement: "If all of the basic feasible solutions are non-degenerate, then the simplex algorithm terminates after a finite number of iterations."

Further, if there exists an optimal solution to the problem, establish that the last iterate is an optimal one.

Question 3

(LP duality)

Consider the linear integer program

$$\begin{aligned} z^*_{IP} := \quad \min_{\boldsymbol{x}} \quad & \mathbf{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}, \\ & \boldsymbol{C}\boldsymbol{x} \le \boldsymbol{d}, \\ & \boldsymbol{x} \in \{0,1\}^n. \end{aligned} \tag{1}$$

Assume that problem (1) is feasible. Let $\{\boldsymbol{x} \mid \boldsymbol{C}\boldsymbol{x} \le \boldsymbol{d},\ \boldsymbol{x} \in \{0,1\}^n\} = \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^N\}$. Consider the Lagrange dual problem

$$\begin{aligned} z^*_{LD} := \quad \max_{\boldsymbol{\mu}} \quad & q(\boldsymbol{\mu}), \\ \text{subject to} \quad & \boldsymbol{\mu} \ge \mathbf{0}, \end{aligned}$$

where

$$\begin{aligned} q(\boldsymbol{\mu}) := \quad \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}), \\ \text{subject to} \quad & \boldsymbol{C}\boldsymbol{x} \le \boldsymbol{d}, \\ & \boldsymbol{x} \in \{0,1\}^n. \end{aligned}$$

(1p) a) Show that z^*_{LD} is the optimal objective value of the following problem

$$\begin{aligned} \max_{y,\boldsymbol{\mu}} \quad & y \\ \text{subject to} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^i + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x}^i - \boldsymbol{b}) \ge y, \quad \forall i = 1, \dots, N \\ & y \in \mathbb{R},\ \boldsymbol{\mu} \ge \mathbf{0} \end{aligned} \tag{2}$$

(1p) b) Show that problem (2) and the following problem have the same optimal objective value

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \in \operatorname{conv}\Big(\big\{\boldsymbol{x} \mid \boldsymbol{C}\boldsymbol{x} \le \boldsymbol{d},\ \boldsymbol{x} \in \{0,1\}^n\big\}\Big) \end{aligned}$$

(1p) c) Let z^*_{LP} denote the optimal objective value of the LP relaxation of (1), with the integrality constraint $\boldsymbol{x} \in \{0,1\}^n$ removed. Show that $z^*_{LP} \le z^*_{LD} \le z^*_{IP}$.

(3p) Question 4

(modelling)

We consider a stepped cantilever beam, which consists of five segments, as shown in Figure 1. Each segment has a rectangular cross-section with width b_i and height h_i to be designed. We assume that each section of the cantilever has the same length l. A vertical load P is applied at a fixed distance L from the support. This load causes the beam to deflect, and induces stress in each segment of the beam with Young's modulus E. Formulate an optimization model to minimize the volume of the beam, subject to constraints on bending stress in all five steps of the beam, to be less than an allowable stress $\sigma_{\max}$; the displacement constraint on the tip deflection to be less than the allowable deflection $\delta_{\max}$, and a specified aspect ration $a_{\max}$ to be maintained between the height and width of beam cross sections.

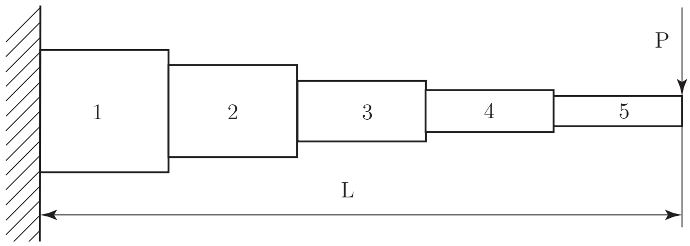

Figure 1: Stepped cantilever beam

Aid: The maximum bending stress at each segment of the beam is

$$\sigma_i = \frac{6PD_i}{b_i h_i^2},$$

where D_i is the maximum distance from the end load. The end deflection can be calculated using Castigliano's second theorem, which states that

$$\delta = \frac{\partial U}{\partial P},$$

where δ is the deflection of the beam, U is the energy stored in the beam due to the applied force P. The energy stored in a cantilever beam is given by

$$U = \int_0^L \frac{P^2 x^2}{2EI} \mathrm{d}x,$$

where I is the area moment of inertia. The moment of inertia of a beam segment with a rectangular cross-section is

$$I_i = \frac{b_i h_i^3}{12}.$$

Question 5

(true or false)

The below three individual claims should be assessed individually. Are they true or false, or is it impossible to say? For each of the three statements, provide an answer, together with a short—but complete— motivation.

(1p) a) Suppose we consider minimizing a function $f \in C^2$ over $\mathbb{R}^n$. Claim: all its stationary points have a positive semi-definite Hessian (i.e., matrix of second-order partial derivatives).

(1p) b) Consider the minimization of a continuous function $f : \mathbb{R}^n \to \Re$ over constraints of the form $g_i(\boldsymbol{x}) \leq 0$, $i = 1, 2, \ldots, m$, defined by the functions $g_i : \mathbb{R}^n \to \Re$. Derive the Lagrangian dual problem for this problem. Claim: the Lagrangian dual problem is a convex one.

(1p) c) Claim: In an optimization problem, a global optimum cannot be a local one.

(3p) Question 6

(optimality conditions)

Prove that $\boldsymbol{x}^* = (1, 1/2, -1)^{\mathrm{T}}$ is optimal for the optimization problem

$$\begin{aligned} \text{minimize} \quad & z = (1/2)\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x} + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{x} + r, \\ \text{subject to} \quad & -1 \leq x_i \leq 1, \quad i = 1, 2, 3, \end{aligned}$$

where

$$\boldsymbol{P} = \begin{pmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} -22.0 \\ -14.5 \\ 13.0 \end{pmatrix}, \quad r = 1.$$

(3p) Question 7

(the basis of the SQP algorithm)

Consider the problem to minimize a function $f \in C^2$ over a set of equality constraints of the form $h_j(\boldsymbol{x}) = 0$, $j = 1, 2, \ldots, \ell$, where all functions h_j also are in C^2. Derive and motivate the subproblem of this algorithm.

Hint: utilize the standard optimality conditions for the original problem.