Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MMG620
OPTIMIZATION, BASIC COURSE

Date:	09–12–14
Time:	House V, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Adam Wojciechowski (0703-088304)
Result announced:	10–01–08 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned} \text{minimize} \quad & z = x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 - 2x_2 \leq -2, \\ & 2x_1 + x_2 \leq 2, \\ & x_1 \in \mathbb{R}, \\ & x_2 \geq 0. \end{aligned}$$

(2p) a) Solve this problem by using phase I and phase II of the simplex method. [*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

(1p) b) Consider the application of the simplex method to a general LP and suppose that you, unlike in the standard procedure taught in this course, at some iteration *a*) choose the entering variable to be a non-basic variable with a negative reduced cost but not having the most negative reduced cost, or *b*) choose the outgoing variable as a basic variable with the $B^{-1}N_{j*}$ component > 0 but not fulfilling the minimum ratio test. Which of these choices is a critical mistake? Motivate *clearly* why that is the case (guessing without a clear motivation will not give you any points).

Question 2

(convexity)

Prove or disprove the following three claims.

(1p) a) $f(x_1, x_2, x_3) = x_1^4 + x_2^2 + 4x_2x_3 + 5x_3^2$ is convex for all $\boldsymbol{x} \in \mathbb{R}^3$.

(1p) b) $f(x_1, x_2) = \max\{2x_1 - x_2, x_2^2\}$ is convex for all $\boldsymbol{x} \in \mathbb{R}^2$.

(1p) c) $f(x_1, x_2) = 2(x_1^3 + x_2^3) + x_1^2 x_2^2 + 4x_2^2$ is convex near $\boldsymbol{x} = (0, 0)^{\mathrm{T}}$ (i.e., there is a small ball around the origin in which the function is convex).

(3p) Question 3

(modeling)

Suppose you want to put up some new wallpaper in your house. You have done some measurements and came to the conclusion that you will need the following pieces of wallpaper: 10 pieces of length 2.7 m, 1 piece of length 2.3 m, 2 pieces of length 2.0 m, 5 pieces of length 1.5 m and 1 piece of length 1.2 m. You can buy the wallpaper in rolls of 10 m length each. Since you have little money left before Christmas, you want to minimize the number of rolls that you have to buy.

Introduce the necessary constants and variables and formulate an integer linear program (i.e., a model that after the relaxation of the integrality constraints is an LP) that will minimize the number of rolls that you need to buy and such that the optimal solution will tell you how the wallpaper rolls should be used (which piece to cut from which roll).

[*Remarks:* Do not solve the model. Make sure that you specify the meaning of each variable and constraint, so that we understand the model.]

(3p) Question 4

(gradient projection)

The gradient projection algorithm is a generalization of the steepest descent method to problems over convex sets. Given a point $\boldsymbol{x}^k$, the next point is obtained according to $\boldsymbol{x}^{k+1} = \mathrm{Proj}_X(\boldsymbol{x}^k - \alpha \nabla f(\boldsymbol{x}^k))$, where X is the convex set over which we minimize, $\alpha > 0$ is the step length, and $\mathrm{Proj}_X(\boldsymbol{y}) = \arg\min_{\boldsymbol{x} \in X} \| \boldsymbol{x} - \boldsymbol{y} \|$ denotes the closest point to $\boldsymbol{y}$ in X.

[*Remark:* if $X = \mathbb{R}^n$, then the method reduces to that of steepest descent.]

Consider the optimization problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := (x_1 + x_2)^2 + 3(x_1 - x_2)^2 \ , \\ \text{subject to} \quad & 0 \le x_1 \le 2, \\ & 1 \le x_2 \le 3. \end{aligned}$$

Start at the point $\boldsymbol{x}^0 = (1\ 2)^{\mathrm{T}}$ and perform two iterations of the gradient projection algorithm using step length $\alpha = 1$. Note that the special form of the feasible region X makes projections very easy! Is the point obtained a global/local optimum? Motivate why/why not!

(3p) Question 5

(strong duality in linear programming)

Consider the following standard form of a linear program:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$. State and prove the Strong Duality Theorem in linear programming.

(3p) Question 6

(the Fritz John conditions)

In the problem to minimize the C^1 function f over the set S, we assume that S is described by differentiable inequality constraints defined by the functions $g_i \in C^1(\mathbb{R}^n)$, $i = 1, \dots, m$, such that

$$S := \{\, \boldsymbol{x} \in \mathbb{R}^n \mid g_i(\boldsymbol{x}) \leq 0, \quad i = 1, \dots, m \,\}.$$

The Fritz John conditions state the following: *If $\boldsymbol{x}^* \in S$ is a local minimum of f over S then there exist multipliers $\mu_0 \in \mathbb{R}$, $\boldsymbol{\mu} \in \mathbb{R}^m$, such that*

$$\mu_0 \nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0}^n, \tag{1a}$$

$$\mu_i g_i(\boldsymbol{x}^*) = 0, \qquad i = 1, \dots, m, \tag{1b}$$

$$\mu_0, \mu_i \geq 0, \qquad i = 1, \dots, m, \tag{1c}$$

$$(\mu_0, \boldsymbol{\mu}^{\mathrm{T}})^{\mathrm{T}} \neq \boldsymbol{0}^{m+1}. \tag{1d}$$

Suppose that we, to the given problem, add the redundant constraint that

$$g_{m+1}(\boldsymbol{x}) := -\frac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}_0\|^2 \leq 0,$$

where the vector $\boldsymbol{x}_0$ is an arbitrary feasible solution.

Establish that if this constraint is added to the problem at hand, then the Fritz John conditions are satisfied at $\boldsymbol{x}_0$.

Draw as many conclusions as can be made regarding the practical usefulness of the Fritz John conditions given the above revelation.

Question 7

(topics in linear programming)

(1p) a) Suppose that a linear program includes a free variable x_j. When transforming this problem into standard form, x_j is replaced by

$$x_j = x_j^+ - x_j^-,$$
$$x_j^+, x_j^- \geq 0.$$

Show that no basic feasible solution can include both x_j^+ and x_j^- as non-zero basic variables.

(1p) b) Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}. \end{aligned} \tag{1}$$

Assume that the objective function vector $\boldsymbol{c}$ cannot be written as a linear combination of the rows of $\boldsymbol{A}$. Show that (**??**) cannot have an optimal solution.

(1p) c) Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned} \tag{P}$$

and the perturbed problem to

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \tilde{\boldsymbol{b}}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n. \end{aligned} \tag{P'}$$

Show that if (P) has an optimal solution, then the perturbed problem (P') cannot be unbounded (independently of $\tilde{\boldsymbol{b}}$).

Good luck!