Def. The (set-theoretic) kernel of \( \varphi : A \to B \) is

\[
\ker \varphi = \{ (x, y) \in A \times A \mid \varphi(x) = \varphi(y) \}.
\]

This is a binary relation on \( A \). It is trivially reflexive and symmetric, and easily seen to be transitive, so it is an equivalence relation. But when \( \varphi \) is a homomorphism, it is even more.

Def. Let \( A \) be an \( \mathcal{L} \)-algebra. A binary relation \( \mathcal{E} = \Delta \times A \) is said to be compatible with \( \mathcal{E} \) if

\[
(x, y_1, \ldots, y_n) \in \mathcal{E} \Rightarrow \left( \delta^A(x, y_1, \ldots, y_n), \delta^A(y_1, \ldots, y_n) \right) \in \mathcal{E}.
\]

An equivalence relation on an \( \mathcal{L} \)-algebra which is compatible with all \( \mathcal{E} \) is said to be an \( \mathcal{L} \)-algebra congruence relation.

Proposition. If \( \varphi : A \to B \) is an \( \mathcal{L} \)-algebra homomorphism, then \( \ker \varphi \) is an \( \mathcal{L} \)-algebra congruence relation on \( A \).

The converse also holds.

Construction. Let \( A \) be an \( \mathcal{L} \)-algebra and \( \mathcal{E} \) a congruence on \( A \). Define the quotient \( A/\mathcal{E} \) to be the set of \( \mathcal{E} \)-equivalence classes, i.e.,

\[
A/\mathcal{E} = \left\{ [x]_\mathcal{E} \mid x \in A \right\}.
\]

Also define the map \( \text{quot}(\mathcal{E}) : A \to A/\mathcal{E} \) as \( x \mapsto [x]_\mathcal{E} \). Then
the quotient \( A/\mathcal{Q} \) is an \( \Omega \)-algebra with operations defined by

\[
\sigma^A (\{x_1\}_{\mathcal{Q}}, \ldots, \{x_{n}\}_{\mathcal{Q}}) = \{\sigma^A (x_1, \ldots, x_{n})\}_{\mathcal{Q}}
\]

for all \( x_1, \ldots, x_{n} \in A \) and \( \sigma \in \Omega \), and \( \text{nat}(\mathcal{Q}) \) is an \( \Omega \)-algebra homomorphism called the natural homomorphism \( A \to A/\mathcal{Q} \) which has \( \text{Ker} \text{nat}(\mathcal{Q}) = \mathcal{Q} \).

Proof. The main thing that needs to be proved is that
the operations on the quotient are well-defined, i.e. that
their values do not depend on which representative \( x_i \) of an
equivalence class is used to compute the value. Concretely this
amounts to the requirement that

\[
\{y_1\}_{\mathcal{Q}} \cdots \{y_{n}\}_{\mathcal{Q}} = \{y_1 y_2 \cdots y_{n}\}_{\mathcal{Q}}
\]

\[
\Rightarrow \{\sigma^A (x_1, \ldots, x_{n})\}_{\mathcal{Q}} = \{\sigma^A (y_1, \ldots, y_{n})\}_{\mathcal{Q}}
\]

for all \( x_1, \ldots, x_{n}, y_1, \ldots, y_{n} \in A \) and \( \sigma \in \Omega \),

but this is just another way of saying \( \mathcal{Q} \) is compatible
with all \( \sigma \in \Omega \), i.e., it is satisfied precisely when \( \mathcal{Q} \) is
a congruence relation!

The rest is trivial. (C) says \( \text{nat}(\mathcal{Q}) \) is an \( \Omega \)-algebra homomorphism
and \( \text{Ker} \text{nat}(\mathcal{Q}) = \{ (x, y) \in A \times A \mid \sigma(x, y) = 1 \} = \mathcal{Q} \).

**First Isomorphism Theorem.** Let \( \phi: A \to B \) be an \( \Omega \)-algebra
epimorphism. Then there is a unique \( \Omega \)-algebra isomorphism
\( \Theta: A/\text{Ker} \phi \to B \) such that \( \phi = \Theta \circ \text{nat}(\text{Ker} \phi) \).
Proof. For the claimed $\Phi = \Theta \circ \text{val}(\ker \Phi)$, the map $\Theta$ would have to satisfy

$$\Theta([x]_{\ker \Phi}) = \phi(x) \text{ for all } x \in A,$$

which clearly defines $\Theta$ on all of $A/\ker \Phi$, and does so universally since $[x]_{\ker \Phi} = [y]_{\ker \Phi} \iff (x,y) \in \ker \Phi \iff \phi(x) = \phi(y).$

Since it defines $\Theta$ on all of $A/\ker \Phi$, there is only one such map satisfying $\Phi = \Theta \circ \text{val}(\ker \Phi)$.

As for $\Theta$ being a homomorphism, we have for every $f \in A$ and $x_1, \ldots, x_n \in A$ that

$$\Theta(fA/\ker \Phi([x_1], \ldots, [x_n])) = \Theta([fA(x_1, \ldots, x_n)]) =$$

$$= \phi(f(x_1, \ldots, x_n)) = f^B(\phi(x_1), \ldots, \phi(x_n)) =$$

$$= f^B(\Theta([x_1]), \ldots, \Theta([x_n])).$$

$\Theta$ is surjective since $\phi$ is surjective. As for injectivity,

$$\Theta([x]) = \Theta([y]) \iff \phi(x) = \phi(y) \iff (x,y) \in \ker \phi \iff [x] = [y].$$

Hence $\Theta$ is an $L_2$-algebra isomorphism.

The consequence of this theorem is that the homomorphisms from $A$ are, up to isomorphism, encoded into the congruences on $A$. This is important because the class of all congruences on $A$ is a set — more precisely, the set of those subsets of $A \times A$ which satisfy the conditions for being a congruence — so the argument is not dependent upon details which differ between different sub-theories.
Enumerating homomorphisms is however only an intermediate goal, primarily we want to construct all $\Omega$-algebras (up to isomorphism, at least). Enumerating the homomorphisms can be used as a tool for this, if we can first find one algebra with homomorphisms to all algebras, since any algebra will then correspond to a congruence on the one algebra. This may seem like a tall order, but there is in fact a straightforward way to deliver upon it, because we have formalised the concepts of signature and $\Omega$-algebra.

In logic, a term is not just an operand of $+$ or $-$, but the generic name for "expression" (in a formalised theory). We denote by $T(\Omega, X)$ (Braden-Nipkow notation) the set of all terms which can be built using symbols in the signature $\Omega$ and "variables" in the set $X$. For example

$$T(\{\times, m(x,y)\}, \{x,y\}) \ni x, y, m(x,y), m(x, y), m(y, x), ...$$

More formally, $T(\Omega, X)$ is inductively defined as the least set satisfying:

(i) $X \subseteq T(\Omega, X)$ and

(ii) if $f \in \Omega$ and $t_1, ..., t_n \in T(\Omega, X)$
     then $f(t_1, ..., t_n) \in T(\Omega, X)$.

Note here that $f(t_1, ..., t_n)$ is not function application but abstract syntax for an expression formed by combining a function symbol $f$ with some terms $t_1, ..., t_n$. We'll return to the matter of how to formalise such things as mathematical objects later.
What can we do with terms? Well, we can evaluate them in an arbitrary $\Sigma$-algebra $B$.

**Lemma.** Let $X$ be a set and $B$ an $\Sigma$-algebra. Then every function $\phi_i : X \to B$ has a unique extension $\phi : T(\Sigma, X) \to B$ such that

$$\phi(f(t_1, \ldots, t_{\omega(\alpha)})) = f^B(\phi(t_1), \ldots, \phi(t_{\omega(\alpha)})) \quad \text{(*)}$$

for all $f \in \Sigma$ and $t_1, \ldots, t_{\omega(\alpha)} \in T(\Sigma, X)$.

**Proof.** Every $f \in T(\Sigma, X) \setminus X$ occurs as the LHS of exactly one instance of (2), so that can be taken as defining $\phi(f)$. \qed

That (*) also has the form of the condition for $\phi$ to be a homomorphism, if merely $T(\Sigma, X)$ is provided with a structure as $\Sigma$-algebra. That structure is

$$f : (t_1, \ldots, t_{\omega(\alpha)}) = f(t_1, \ldots, t_{\omega(\alpha)})$$

for all $t_1, \ldots, t_{\omega(\alpha)} \in T(\Sigma, X)$.

It is called the **free $\Sigma$-algebra** on $X$.

**Theorem.** Let $A$ be an $\Sigma$-algebra generated by $X \subseteq A$. Then $A$ is isomorphic to a quotient of $T(\Sigma, X)$.

Consequently, any $\Sigma$-algebra can be constructed as a quotient of a free $\Sigma$-algebra.