FOURIER ANALYSIS & METHODS 2020.02.17

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. The illustrious Fourier transform

The following is a useful and fundamental collection of facts about the Fourier transform. It may be useful to introduce the notations

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \hat{f}(\xi).$$

Sometimes we feel like a wide hat, sometimes a narrow hat, and sometimes we need that big $\mathcal{F}$. It is useful to be fluent with all three equivalent notations.

Theorem 1 (Properties of the Fourier transform)**.** *Assume that everything below is well defined. Then, the Fourier transform,*

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$$

satisfies

(1) $\mathcal{F}(f(x-a))(\xi) = e^{-ia}\hat{f}(\xi)$.
(2) $\mathcal{F}(f')(\xi) = i\xi\hat{f}(\xi)$
(3) $\mathcal{F}(xf(x))(\xi) = i\mathcal{F}(f)'(\xi)$
(4) $\mathcal{F}(f * g)(\xi) = \hat{f}(\xi)\hat{g}(\xi)$

Proof: We just compute (we are being a bit naughty, not bothering with issues of convergence, but all such issues are indeed rigorously verifiable, so not to worry). First

$$\mathcal{F}(f(x-a))(\xi) = \int_{\mathbb{R}} f(x-a)e^{-ix\xi}dx.$$

Change variables. Let $t = x - a$, then $dt = dx$, and $x = t + a$ so

$$\mathcal{F}(f(x-a))(\xi) = \int_{\mathbb{R}} f(t)e^{-i(t+a)\xi}dt = e^{-ia\xi}\hat{f}(\xi).$$

The next one will come from integrating by parts:

$$\int_{\mathbb{R}} f'(x)e^{-ix\xi}dx = f(x)e^{-ix\xi}\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} -i\xi f(x)e^{-ix\xi}dx = i\xi\hat{f}(\xi).$$

The boundary terms vanish because of reasons (again it is $\mathcal{L}^1$ and $\mathcal{L}^2$ theory stuff). Similarly we compute

$$\int_{\mathbb{R}} xf(x)e^{-ix\xi}dx = -\frac{1}{i}\int_{\mathbb{R}} f(x)\frac{d}{d\xi}e^{-ix\xi}dx = i\frac{d}{d\xi}\int_{\mathbb{R}} f(x)e^{-ix\xi}dx = i\mathcal{F}(f)'(\xi).$$

Finally,

$$\mathcal{F}(f*g)(\xi) = \int_{\mathbb{R}} f*g(x)e^{-ix\xi}dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)e^{-ix\xi}dydx.$$

We do a little sneaky trick

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)e^{-ix\xi}e^{-iy\xi}e^{iy\xi}dydx$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)e^{-i(x-y)\xi}g(y)e^{-iy\xi}dydx.$$

Let $z = x-y$. Then $dz = -dy$ so

$$= \int_{\mathbb{R}}\int_{\infty}^{-\infty} f(z)e^{-iz\xi}(-dz)g(y)e^{-iy\xi}dy = \int_{\mathbb{R}}\int_{\mathbb{R}} f(z)e^{-iz\xi}dzg(y)e^{-iy\xi}dy$$

$$= \hat{f}(\xi)\hat{g}(\xi).$$

It shall be quite useful to know how to "undo" the Fourier transform.

Theorem 2 (Extension of Fourier transform to $\mathcal{L}^2$)**.** *There is a well defined unique extension of the Fourier transform to $\mathcal{L}^2(\mathbb{R})$. The Fourier transform of an element of $\mathcal{L}^2(\mathbb{R})$ is again an element of $\mathcal{L}^2(\mathbb{R})$. Moreover, for any $f \in \mathcal{L}^2(\mathbb{R})$ we have the FIT (Fourier Inversion Theorem):*

eq:fit (1.1)
$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{ix\xi}d\xi.$$

The theory item FIT is a *Julklapp.* All you need to know is the equation (eq:fit 1.1). The next theorem is also a theory item, with a short proof. The key is to start on the right side and use the FIT.

Theorem 3 (Plancharel)**.** *For any $f \in \mathcal{L}^2(\mathbb{R})$, $\hat{f} \in \mathcal{L}^2(\mathbb{R})$. Moreover,*

$$\langle \hat{f}, \hat{g}\rangle = 2\pi\langle f, g\rangle,$$

and thus

$$||\hat{f}||^2_{\mathcal{L}^2} = 2\pi||f||^2,$$

for all f and g in $\mathcal{L}^2(\mathbb{R})$.

Proof: Start with the right side and use the FIT on f, to write

$$2\pi\langle f, g\rangle = 2\pi\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2\pi}e^{ix\xi}\hat{f}(\xi)\overline{g(x)}d\xi dx = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{ix\xi}\hat{f}(\xi)\overline{g(x)}d\xi dx.$$

Move the complex conjugate to engulf the $e^{ix\xi}$,

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\overline{g(x)e^{-ix\xi}}d\xi dx.$$

Swap the order of integration and integrate x first:

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\overline{g(x)e^{-ix\xi}}dxd\xi = \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi = \langle \hat{f}, \hat{g}\rangle.$$

We may from time to time use the following cute fact as well.

Lemma 4 (Riemann & Lebesgue)**.** *Assume* $f \in \mathcal{L}^1(\mathbb{R})$*. Then,*

$$\lim_{\xi \to \pm\infty} \hat{f}(\xi) = 0.$$

We shall indeed need to actually prove the next one, because it's going to be quite important for solving the heat equation on the real line.

1.1. **The big bad convolution approximation theorem.** This theory item is Theorem 7.3, regarding approximation of a function by convoluting it with a so-called "approximate identity." This theorem and its proof are both rather long. The proof relies very heavily on knowing the definition of limits and how to work with those definitions, so if you're not comfortable with ϵ and δ style arguments, it would be advisable to brush up on these.

Theorem 5. *Let* $g \in L^1(\mathbb{R})$ *such that*

$$\int_{\mathbb{R}} g(x)dx = 1.$$

Define

$$\alpha = \int_{-\infty}^{0} g(x)dx, \quad \beta = \int_{0}^{\infty} g(x)dx.$$

Assume that f *is piecewise continuous on* $\mathbb{R}$ *and its left and right sided limits exist for all points of* $\mathbb{R}$*. Assume that either* f *is bounded on* $\mathbb{R}$ *or that* g *vanishes outside of a bounded interval. Let, for* $\varepsilon > 0$*,*

$$g_\epsilon(x) = \frac{g(x/\epsilon)}{\epsilon}.$$

Then

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \forall x \in \mathbb{R}.$$

Proof. **Idea 1:** Do manipulations to get a "left side" statement and a "right side" statement.

We would like to show that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y)dy = \alpha f(x+) + \beta f(x-)$$

which is equivalent to showing that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y)dy - \alpha f(x+) - \beta f(x-) = 0.$$

We now insert the definitions of α and β, so we want to show that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy - \int_{0}^{\infty} f(x-)g(y)dy = 0.$$

We can prove this if we show that

$$\heartsuit : \lim_{\varepsilon \to 0} \int_{-\infty} f(x-y) g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy = 0$$

and also

$$\star : \lim_{\varepsilon \to 0} \int_0^\infty f(x-y) g_\varepsilon(y) dy - \int_0^\infty f(x-) g(y) dy = 0.$$

In the textbook, Folland proves that $\star$ holds. So, for the sake of diversity, we prove that $\heartsuit$ holds. The argument is the same for both, so proving one of them is sufficient.

Hence, we would like to show that by choosing ε sufficiently small, we can make

$$\int_{-\infty}^0 f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^0 f(x+) g(y) dy$$

as small as we like. To make this precise, let us assume that "as small as we like" is quantified by a very small $\delta > 0$. Then we show that for sufficiently small ε we obtain

$$\left| \int_{-\infty}^0 f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^0 f(x+) g(y) dy \right| < \delta.$$

Idea 2: Smash the two integrals together:

$$\int_{-\infty}^0 \left(f(x-y) g_\varepsilon(y) - f(x+) g(y) \right) dy.$$

Well, this is a bit inconvenient, because in the first part we have g_ε, but in the second part it's just g.

Idea 3: Sneak g_ε into the second term. We make a small observation,

$$\int_{-\infty}^0 g(y) dy = \int_{-\infty}^0 g(z/\varepsilon) \frac{dz}{\varepsilon} = \int_{-\infty}^0 g_\varepsilon(z) dz$$

Above, we have made the substitution $z = \varepsilon y$, so $y = z/\varepsilon$, and $dz/\varepsilon = dy$. The limits of integration don't change. By this calculation,

$$\int_{-\infty}^0 f(x+) g(y) dy = \int_{-\infty}^0 f(x+) g_\varepsilon(y) dy.$$

(Above the integration variable was called z, but what's in a name? The name of the integration variable doesn't matter!). Moreover, note that $f(x+)$ is a constant, so it's just sitting there doing nothing. Hence, we have computed that

$$\int_{-\infty}^0 \left(f(x-y) g_\varepsilon(y) - f(x+) g(y) \right) dy = \int_{-\infty}^0 g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy.$$

Remember that $y \leq 0$ where we're integrating. Therefore, $x - y \geq x$.

Idea 4: Use the definition of right hand limit:

$$\lim_{y \uparrow 0} f(x-y) = f(x+) \implies \lim_{y \uparrow 0} f(x-y) - f(x+) = 0.$$

By the definition of limit there exists $y_0 < 0$ such that for all $y \in (y_0, 0)$

$$|f(x-y) - f(x+)| < \widetilde{\delta}.$$

We are using $\widetilde{\delta}$ for now, to indicate that $\widetilde{\delta}$ is going to be something in terms of δ, engineered in such a way that at the end of our argument we get that for ε sufficiently small,

$$\left| \int_{-\infty}^0 g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy \right| < \delta.$$

To figure out this $\widetilde{\delta}$, we use our estimate on the part of the integral from y_0 to 0,

$$\left|\int_{y_0}^{0}(f(x-y)-f(x+))g_\varepsilon(y)dy\right| \le \int_{y_0}^{0}|f(x-y)-f(x+)||g_\varepsilon(y)|dy$$

$$\le \widetilde{\delta}\int_{y_0}^{0}|g_\varepsilon(y)|dy \le \widetilde{\delta}\int_{\mathbb{R}}|g_\varepsilon(y)|dy = \widetilde{\delta}||g||.$$

Above, we have used the same substitution trick to see that

$$\int_{\mathbb{R}}|g_\varepsilon(y)|dy = \int_{\mathbb{R}}|g(z)|dz = ||g||,$$

where $||g||$ is the $L^1(\mathbb{R})$ norm of g. By assumption, $g \in L^1(\mathbb{R})$, so this L^1 norm is finite. Moreover, because we know that

$$\int_{\mathbb{R}} g(y)dy = 1,$$

we know that

$$||g|| = \int_{\mathbb{R}}|g(y)|dy \ge \left|\int_{\mathbb{R}} g(y)dy\right| = 1.$$

So, let

$$\widetilde{\delta} = \frac{\delta}{2||g||}.$$

Note that we're not dividing by zero, by the above observation that $||g|| \ge 1$. So, this is a perfectly decent number. Then, we have the estimate (repeating the above estimate)

$$\left|\int_{y_0}^{0}(f(x-y)-f(x+))g_\varepsilon(y)dy\right| \le \int_{y_0}^{0}|f(x-y)-f(x+)||g_\varepsilon(y)|dy$$

$$\le \widetilde{\delta}\int_{y_0}^{0}|g_\varepsilon(y)|dy \le \widetilde{\delta}\int_{\mathbb{R}}|g_\varepsilon(y)|dy = \widetilde{\delta}||g|| = \frac{\delta}{2}.$$

Idea 5: To deal with the other part of the integral, from $-\infty$ to y_0, consider the two cases given in the statement of the theorem separately. It is important to remember that

$$y_0 < 0.$$

So, we wish to estimate

$$\left|\int_{-\infty}^{y_0}(f(x-y)-f(x+))g_\varepsilon(y)dy\right|.$$

First, let us assume that f is bounded, which means that there exists $M > 0$ such that $|f(x)| \le M$ holds for all $x \in \mathbb{R}$. Hence

$$|f(x-y)-f(x+)| \le |f(x-y)| + |f(x+)| \le 2M.$$

So, we have the estimate

$$\left|\int_{-\infty}^{y_0}(f(x-y)-f(x+))g_\varepsilon(y)dy\right| \le \int_{-\infty}^{y_0}|f(x-y)-f(x+)||g_\varepsilon(y)|dy \le 2M\int_{-\infty}^{y_0}|g_\varepsilon(y)|dy.$$

We shall do a substitution now, letting $z = y/\varepsilon$. Then, as we have computed before,

$$\int_{-\infty}^{y_0}|g_\varepsilon(y)|dy = \int_{-\infty}^{y_0/\varepsilon}|g(z)|dz.$$

Here the limits of integration do change, because $y_0 < 0$. Specifically $y_0 \neq 0$, which is why the top limit changes. We're integrating between $-\infty$ and y_0/ε. We know that $y_0 < 0$. So, when we divide it by a really small, but still positive number, like ε, then $y_0/\varepsilon \to -\infty$ as $\varepsilon \to 0$. Moreover, we know that

$$\int_{-\infty}^{0} |g(y)|dy < \infty.$$

What this really means is that

$$\lim_{R\to-\infty} \int_{R}^{0} |g(y)|dy = \int_{-\infty}^{0} |g(y)|dy < \infty.$$

Hence,

$$\lim_{R\to-\infty} \int_{-\infty}^{0} |g(y)|dy - \int_{R}^{0} |g(y)|dy = 0.$$

Of course, we know what happens when we subtract the integral, which shows that

$$\lim_{R\to-\infty} \int_{-\infty}^{R} |g(y)|dy = 0.$$

Since

$$\lim_{\varepsilon\to 0} y_0/\varepsilon = -\infty,$$

this shows that

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{y_0/\varepsilon} |g(y)|dy = 0.$$

Hence, by definition of limit there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{-\infty}^{y_0/\varepsilon} |g(y)|dy < \frac{\delta}{4(M+1)}.$$

Then, combining this with our estimates, above, which we repeat here,

$$\left|\int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy\right| \le \int_{-\infty}^{y_0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy \le 2M \int_{-\infty}^{y_0} |g_\varepsilon(y)|dy$$
$$< 2M\frac{\delta}{4(M+1)} < \frac{\delta}{2}.$$

Therefore, we have the estimate that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\left|\int_{-\infty}^{0} g_\varepsilon(y)\left(f(x-y) - f(x+)\right)dy\right|$$
$$\le \int_{-\infty}^{0} |g_\varepsilon(y)||f(x-y)-f(x+)|dy \le \int_{-\infty}^{y_0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy + \int_{y_0}^{0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy$$
$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Finally, we consider the other case in the theorem, which is that g vanishes outside a bounded interval. We retain the first part of our estimate, that is

$$\int_{y_0}^{0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy < \frac{\delta}{2}.$$

Next, we again observe that

$$\lim_{\varepsilon\downarrow 0} \frac{y_0}{\varepsilon} = -\infty.$$

By assumption, we know that there exists some $R > 0$ such that

$$g(x) = 0 \forall x \in \mathbb{R} \text{ with } |x| > R.$$

Hence, we may choose ε sufficient small so that

$$\frac{y_0}{\varepsilon} < -R.$$

Specifically, let

$$\varepsilon_0 = \frac{1}{-Ry_0} > 0.$$

Then for all $\varepsilon \in (0, \varepsilon_0)$ we compute that

$$\frac{y_0}{\varepsilon} < -R.$$

Hence for all $y \in (-\infty, y_0/\varepsilon)$ we have $g(y) = 0$. Thus, we compute as before using the substitution $z = y/\varepsilon$,

$$\int_{-\infty}^{y_0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy = \int_{-\infty}^{y_0/\varepsilon} |f(x - \varepsilon z) - f(x+)||g(z)|dz = 0,$$

because $g(z) = 0 \forall z \in (-\infty, y_0/\varepsilon)$. Thus, we have the total estimate that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\left| \int_{-\infty}^{0} g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy \right|$$

$$\leq \int_{-\infty}^{0} |g_\varepsilon(y)||f(x-y)-f(x+)|dy \leq \int_{-\infty}^{y_0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy + \int_{y_0}^{0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy$$

$$< 0 + \frac{\delta}{2} \leq \delta.$$

□

1.2. **Exercises for the week to be demonstrated.** On Monday in the large group we shall have:

(1) (7.2.13.b) Use Plancharel's theorem to compute:

$$\int_{\mathbb{R}} \frac{t^2}{(t^2+a^2)(t^2+b^2)} dt = \frac{\pi}{a+b}.$$

(2) (Eö 12) Let

$$f(t) = \int_0^1 \sqrt{w} e^{w^2} \cos(wt) dw.$$

Compute

$$\int_{\mathbb{R}} |f'(t)|^2 dt.$$

(3) (7.4.1.a,b) Compute the Fourier sine and cosine transforms of e^{-kx}. These are defined, respectively, to be

$$\mathcal{F}_s[f](\xi) = \int_0^\infty f(x) \sin(\xi x) dx, \quad \mathcal{F}_c[f](\xi) = \int_0^\infty f(x) \cos(\xi x) dx.$$

On Wednesday or Friday depending on your group we shall have:

(1) (Eö 6.a, b) Compute the Fourier transforms of:

$$\frac{t}{(t^2+a^2)^2}, \quad \frac{1}{(t^2+a^2)^2}.$$

(2) (Eö 7) A function has Fourier transform
$$\hat{f}(\xi) = \frac{\xi}{1+\xi^4}.$$
Compute
$$\int_{\mathbb{R}} t f(t) dt, \quad f'(0).$$
(3) (7.3.2) Use the Fourier transform to derive the solution of the inhomogeneous heat equation $u_t = k u_{xx} + G(x,t)$ with initial condition $u(x,0) = f(x)$ (assume $f \in \mathcal{L}^2(\mathbb{R})$:
$$u(x,t) = f * K_t(x) + \int_{\mathbb{R}} \int_0^t G(y,s) K_{t-s}(x-y) ds dy.$$
Here
$$K_t(x) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/4kt}.$$

1.3. **Exercises for the week to be done oneself.**

(1) (Eö 9) Compute (with help of Fourier transform)
$$\int_{\mathbb{R}} \frac{\sin(x)}{x(x^2+1)} dx.$$
(2) (Eö 67) Compute the Fourier transform of the characteristic function for the interval (a,b) both directly and by using the known case for the interval $(-a,a)$.
(3) (7.2.8) Given $a > 0$ let $f(x) = e^{-x} x^{a-1}$ for $x > 0$, $f(x) = 0$ for $x \le 0$. Show that $\hat{f}(\xi) = \Gamma(a)(1+i\xi)^{-a}$ where Γ is the Gamma function.
(4) (7.2.12) For $a > 0$ let
$$f_a(x) = \frac{a}{\pi(x^2+a^2)}, \quad g_a(x) = \frac{\sin(ax)}{\pi x}.$$
Use the Fourier transform to show that: $f_a * f_b = f_{a+b}$ and $g_a * g_b = g_{\min(a,b)}$.
(5) (Eö 6.d,e) Compute the Fourier transform of:
$$e^{-a|t|} \sin(bt), \quad (a, b > 0), \quad \frac{t}{t^2+2t+5}.$$
(6) (Eö 15) Find a solution to the equation
$$u(t) + \int_{-\infty}^t e^{\tau - t} u(\tau) d\tau = e^{-2|t|}.$$
(7) (Eö 11) For the function
$$f(t) = \int_0^2 \frac{\sqrt{w}}{1+w} e^{iwt} dw,$$
compute
$$\int_{\mathbb{R}} f(t) \cos(t) dt, \quad \int_{\mathbb{R}} |f(t)|^2 dt.$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).