TMA947 / MMG621 — Nonlinear optimisation

Exercise 5 – LP duality and sensitivity analysis, Subgradient optimization

October 23, 2017

E5.1 (easy) Formulate the dual to the following problem

$$\begin{aligned} \text{minimize} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 3 \\ & x_1 + x_2 \leq 10 \\ & 5x_1 - x_2 \geq 8 \\ & x_1 \geq 0, \\ & x_2 \leq 0. \end{aligned}$$

E5.2 (medium)

(a) If an LP primal is infeasible, what can you say about its LP dual?

(b) If the LP primal has an optimal solution with reduced costs strictly greater than zero. What can you say about its LP dual?

(c) If the LP dual is unbounded, what can you say about the LP primal?

(d) According to theorem 10.15 an optimal primal dual pair must satisfy primal feasiblity, dual feasibility and complementarity. Which of these conditions is satisfied during the iteration of the simplex algorithm?

E5.3 (easy) Consider the following LP problem

$$\begin{aligned} \text{minimize} \quad & 9x_1 + 3x_2 + 2x_3 + 2x_4 \\ \text{subject to} \quad & \sum_{i=1}^{4} x_i \geq 1, \\ & 3x_1 - x_2 + 2x_4 \geq 1, \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \leq 0. \end{aligned}$$

(a) Use LP duality and a graphical solution to obtain the optimal objective value z^*.

(b) Use complementary slackness to obtain the optimal solution $\boldsymbol{x}^*$.

E5.4 (medium) Let the constraint matrix

$$A = \begin{pmatrix} \cdots \boldsymbol{a}_1^T \cdots \\ \cdots \boldsymbol{a}_2^T \cdots \\ \vdots \\ \cdots \boldsymbol{a}_n^T \cdots \end{pmatrix}.$$

Consider the relaxation of a standard LP problem $\min\{\boldsymbol{c}^T\boldsymbol{x} | A\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$, where we allow a violation of the constraints, but bound the sum of violations by epsilon.

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{a}_i^T\boldsymbol{x} \geq b_i - v_i,\ i = 1, \dots, n, \\ & \sum_{i=1}^{n} v_i \leq \varepsilon, \\ & \boldsymbol{x} \geq 0, \boldsymbol{v} \geq 0. \end{array}$$

Formulate the dual and give it an interpretation.

E5.5 (easy) Consider the following linear program.

$$\begin{array}{lrcl} \text{minimize} & z = -2x_1 + x_2 & & \\ \text{subject to} & x_1 - 3x_2 & \leq & \beta, \\ & 0 \leq x_1, & & \\ & 0 \leq x_2 & \leq & 2. \end{array}$$

Assuming that $\beta \leq 0$, we rewrite the problem on standard form.

$$\begin{array}{lrcl} \text{minimize} & z = -2x_1 + x_2, & & \\ \text{subject to} & -x_1 + 3x_2 - s_1 & = & -\beta, \\ & x_2 + s_2 & = & 2, \\ & x_1, \quad x_2, \quad s_1, \quad s_2 & \geq & 0. \end{array}$$

The optimal solution for $\beta = -3$ has x_1 and x_2 as basis variables. What is the marginal change in optimal objective when β is varied from its current value of -3 (i.e. calculate $\frac{\partial z^*}{\partial \beta}$).

E5.6 (easy) Consider exercise **E4.8**. State for which values of the first component (the one which is 7 now) of the right-hand side vector the optimal basis remains being the optimal one.

E5.7 (easy) Consider again exercise **E4.8**.

(a) Assume that the cost coefficient of x_1 is modfied to $2 + \varepsilon$. State for which values of ε the current optimal basis remains being the optimal one.

(b) Assume that the cost coefficient of x_2 is modfied to $-1 + \varepsilon$. State for which values of ε the current optimal basis remains being the optimal one.

(c) Assume that the cost coefficient of x_3 is modfied to $1 + \varepsilon$. State for which values of ε the current optimal basis remains being the optimal one.

E5.8 (easy) We assume that $\mathbf{x}^* = (\mathbf{x}_B^{\mathrm{T}}, \mathbf{x})_N^{\mathrm{T}} = ((\mathbf{B}^{-1}\mathbf{b})^{\mathrm{T}}, (\mathbf{0}^{n-m})^{\mathrm{T}})$ is an optimal basic feasible solution to a linear program with

$$\mathbf{B} = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}.$$

The right hand side of the linear program $\mathbf{b}$ was modified to $\mathbf{b} + \varepsilon \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Find ε to maintain the optimality of $\mathbf{x}^*$.

E5.9 (easy) The function $f(x)$ is explicitly stated as

$$f(x) = \begin{cases} x, & 0 \leq x \leq 2, \\ 4 - x, & 2 \leq x \leq 4. \end{cases}$$

(a) Find the subdifferential of $f(x)$ at $x = 1$.

(b) Find the subdifferential of $f(x)$ at $x = 2$.

E5.10 (easy) Consider Example 6.16 in the textbook. There, the dual function q is explicitely stated as

$$q(\mu) = \begin{cases} -3 + 5\mu, & 0 \leq \mu \leq 1/4, \\ -2 + \mu, & 1/4 \leq \mu \leq 1/2, \\ -3\mu, & 1/2 \leq \mu. \end{cases}$$

(a) Find the subdifferential $\partial q(\mu)$ at $\mu = 1/8$.

(b) Find the subdifferential $\partial q(\mu)$ at $\mu = 1/2$. (Note: Use Proposition 6.20d)

E5.11 (easy) Consider the unconstrained optimization problem

$$\begin{aligned} f^* = \text{minimum} \quad & f(\boldsymbol{x}), \\ \text{subject to} \quad & \boldsymbol{x} \in \mathbb{R}^2. \end{aligned}$$

where f is a convex function.

(a) At the point $\bar{\boldsymbol{x}} = (2, 1)^T$, we have that $f(\bar{\boldsymbol{x}}) = 2$ and that $\boldsymbol{g} = (1, -1)^T$ is a subgradient to f at $\bar{\boldsymbol{x}}$. What can you say about f^* and $\boldsymbol{x}^*$?

(b) At the point $\tilde{\boldsymbol{x}} = (0, 1)^T$, we have that $f(\tilde{\boldsymbol{x}}) = -1$ and that $\boldsymbol{g}^1 = (-1, 0)^T$, $\boldsymbol{g}^2 = (1, 2)^T$ and $\boldsymbol{g}^3 = (1, -1)^T$ are subgradients to f at $\tilde{\boldsymbol{x}}$. What can you say about f^* and $\boldsymbol{x}^*$?

E5.12 (medium) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Show that for any $\boldsymbol{x} \in \mathbb{R}^n$, the subdifferential $\partial f(\boldsymbol{x})$ is a convex set.

E5.13 (medium) Consider the problem

$$\begin{aligned} \min \quad & \|\mathbf{x}\|_1 \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{aligned}$$

where the variable is $\mathbf{x} \in \mathbb{R}^n$, and the data are $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. We assume that $\mathbf{A}$ has full rank, i.e., $m < n$ and rank $\mathbf{A} = m$. Write down the projected subgradient update (6.40) for this specific problem. Use (12.44) to find the projection.