1 User Guide to hunting positively - invariant sets and ω - limit sets.

We consider flows or dynamical systems corresponding to autonomous differential equations

$$\dot{x} = f(x), \quad f : G \rightarrow \mathbb{R}^N$$

A system has naturally many positively - invariant sets, for example the whole domain G is always an positively - invariant set, but it is not very interesting. We like to find possibly narrow invariant sets showing more precisely where trajectories or solutions to the equation tend when t tends to the upper bound of the maximal time interval (usually $t \rightarrow \infty$ if the trajectory is bounded and has compact closure).

A general idea that is used to answer many questions about behaviour of solutions (trajectories) of the equations, is the idea of test functions. One checks if the velocities $f(x)$ are directed inside or outside with respect to the sets like $Q = \{x \in U : V(x) \leq C\}$ or $Q = \{x \in U : V(x) \geq C\}$ defined by some simple test functions $V : U \rightarrow \mathbb{R}$, $U \subset G$. A more refined variant of this idea by Lyapunov is to find test a function that is monotone along the trajectories $\varphi(t, \xi)$ of the equation. The advantage of the idea with test functions is that one does not need to solve the equation to use it.

How to find an positively **- invariant set?**

Method 1. We find a test function $V(x)$ that has some level sets $\partial Q = \{x : V(x) = C\}$ that are closed curves (or surfaces in higher dimensions) enclosing a bounded domain Q. Typical examples are $V(x, y) = x^2 + y^2 = R^2$ - circle or radius R, or $V(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ - ellipse, or more complicated ones as $V(x, y) = x^6 + ay^4$ - smoothed rectangle shape or squeezed ellipse, $V(x, y) = x^2 + xy + y^2 = C$ - ellipse rotated in $\pi/4$ and having axes A and B related as $A/B = \sqrt{3}$etc.

- To show that a particular level set ∂Q bounds an positively - invariant set Q we check the sign of the directional derivative of V along the velocity in the equation: $V_f(x) = (\nabla V \cdot f)(x)$ for all points on the level set $\{V(x) = C\}$ for a particular constant C.
- The sign of $V_f(x)$ shows if the trajectories go to the same side of the level set as the gradient ∇V (if $V(x) > 0$) or to the opposite side (if $V_f(x) < 0$).
- Then if $V(x)$ is rising for x going out of Q, and $V_f(x) < 0$ then the domain Q inside this level set ∂Q (curve in the plane case) will be positively - invariant. Similarly if $V(x)$ is decreasing out of this level set, and $V_f(x) < 0$ on the level set ∂Q then the domain Q inside this level set will be positively - invariant.

In the opposite case the complement to Q that is $\mathbb{R}^N \backslash Q$ will be positively - invariant and trajectories $\varphi(t, \xi)$ starting in this complement: $\xi \in \mathbb{R}^N \backslash Q$ will never enter Q.

First integrals. A very particular case of test functions are functions that are constant on all trajectories $\varphi(t, \xi)$ of the system. It means that $\frac{d}{dt} V(\varphi(t, \xi)) = V_f(x) \equiv 0$. Usually but not always, such test functions have the meaning of the total energy in the system. In this case all level sets of the first integral are invariant sets, because velocities $f(x)$ are tangent vectors to the level sets in this case.

Method 2. If it is difficult to guess a simple test function giving one closed formula for the boundary of an positively - invariant set as in the Method 1, then one can try to identify a boundary for an positively

- invariant set as a curve (or a surface in higher dimensions) consisting of a number of simple peaces, for example straight segments.

The simplest positively - invariant set of such kind would be a rectangle (a rectangular box in higher dimensions) with sides parallel to coordinate axes. Then to check that this rectangle is an positively - invariant one needs just to check the sign of x or y - components of $f(x)$ on these segments, showing that trajectories go inside or outside of the rectangle.

Application to Poincare Bendixson theorem

One searchs often positively - invariant sets with special properties. For example to apply the Poincare-Bendixson theorem one needs to find an positively - invariant set without equilibrium points. On the other hand it is known that any periodic orbit in plane encloses at least one equilibrium point. It means that a typical positively - invariant set for applying the Poincare-Bendixson theorem should be ring shaped with at least one hole in the middle including a repelling non stable equilibrium point.

Check list for application of the Poincare-Bendixson theorem.

• One starts with applying one of the two methods above to find a compact positively - invariant set Q with at least one equilibrium point inside it. Such set Q does not satisfy conditions in the Poincare-Bendixson theorem yet.

• To identify holes around the equilibriums in the middle (one must find all such equilibrium points at the end !), one needs often to find one more test function for each of them, to show that trajectories do not enter a neighbourhood of each of the equilibriums.

• Alternatively one can use the linearization to show that this equilibrium is repeller and therefore trajectories cannot enter some small neighbourhood of the equilibrium in the middle of the set Q.

• One must check at the end that the found positively invariant annulus (closed ring shaped domain) does not include equilibrium points (not at the boundary either!) It is often simpler to do after carrying out estimates for V_f.

How to find an ω - limit set?

ω - limit sets live naturally inside ω - invariant sets. In case one can find a very small ω - invarinat set the position and the size of the ω - limit set inside it will be rather well defined.

Description properties of ω - limit sets is the main and the most complicated problem in the theory of dynamical systems. Even numerical investigation of limit sets in dimension higher then 2 is rather complicated and needs advanced mathematical tools.

In autonomous systems the plane $\mathbb{R}^2$ limit sets can be only of three types: a) **equilibrium points, b) periodic orbits, and c) closed curves consisting of finite number of equilibrium points connected by open orbits.** It is an extension of the Poincare-Bendixson theorem.

The analytic identification or at least effective localization of ω - limit sets is possible with help of La Salle's invariance theorem. It states that ω - limit sets are subsets of zero level sets of $V_f(x) = (\nabla V \cdot f)(x)$ for appropriate Lyapunov functions $V(x)$ satisfying $V_f(x) \leq 0$.

This theorem helps in particular to find ω - limit sets that are asymptotically stable equilibrium points, by a rather simple checking the behaviour of the velocity $f(x)$ on the zero level set where $V_f(x) = 0$.

One can also investigate asymptotically stable equilibrium points with help of so called "strong" Lyapunov functions that satisfy the strict inequality $V_f(x) < 0$ for $x \neq 0$.

It is difficult in practice to find analytically ω - limit sets of two other types. It is possible if one can find analytically a zero level set $V_f^{-1}(0)$ that is a closed curve in plane. Then this level set belongs to one of the two other types: periodic orbit or a chain of equilibrium points connected by open orbits.

Such an analytic construction is not known for the equation with periodic orbit in the second home assignment, despite the fact that special techniques were developed to show that the periodic orbit there is unique.

If a system has first integrals, then level sets of first integrals give a good tool to identify ω - limit sets because these level sets consist of orbits and are very narrow invariant sets themself. The existence of first integrals is usually a sign that energy of the system is preserved, that is a rather special situation.

The observations above show that in many practical situations we can find ω - limit sets that are asymptotically stable equilibrium points.

For systems in plane we can with help of Poincare Bendixson theorem also show that in certain situations ω - limit sets are periodic orbits but cannot give a formula for them and cannot state how many they are.

ω - limit sets in the plane that are more complicated then equilibrium points, is possible to describe analytically in the case when for a Lyapunov function $V(x)$ the zero level set $V_f^{-1}(0)$ is a closed curve in the plane and the corresponding equation can be investigated analytically.