Class Lectures (for Chapter 9)

Total Variation

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Remarks:

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We say f **is of bounded variation** on $[a, b]$ if $TV_{[a,b]}(f) < \infty$; otherwise we say f **is of unbounded variation** on $[a, b]$.

Remarks:
(i). If f is monotone increasing, then $TV_{[a,b]}(f) = f(b) - f(a)$.
(ii). $TV_{[a,b]}(-f) = TV_{[a,b]}(f)$.
(iii). If f is the indicator function of the rationals, then f is of unbounded variation on every (nontrivial) interval.

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$$= g(b) - g(a) + h(b) - h(a)$$

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The two summands are increasing in x by Step 2, where for the second term we also use the fact that $TV_{[0,x]}(-f) = TV_{[0,x]}(f)$.
QED

Signed measures and function of finite Variation

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There is a 1-1 correspondence between signed measures and functions of bounded variation. The bijection is given by μ a signed measure on $[0, 1]$ is sent to the bounded variation function

$$F_\mu(x) := \mu([0, x].$$

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The Cantor Ternary function is *not* absolutely continuous.

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First, note that f is continuous if and only if μ_f has no atoms. If these equivalent conditions fail, then both sides in the proposition fail. Hence we can assume that f is continuous or equivalently μ_f is continuous (i.e. no atoms).

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implying that

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which is equivalent to

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If f is absolutely continuous, then $\mu_f \ll m$.
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and so $\mu_f(\bigcup_{i=1}^{N}(a_i, b_i)) < \epsilon$. By letting $N \to \infty$, we have $\mu_f(U) \leq \epsilon$. Since $A \subseteq U$, this gives $\mu_f(A) \leq \epsilon$ and since ϵ is arbitrary, we get $\mu_f(A) = 0$.

Finite total variation and absolute continuity

Proposition If $f : [0, 1] \to R$ is absolutely continuous, then it has bounded variation.

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$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

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QED (Recall the Cantor Ternary function)

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Note that for the Cantor Ternary function, the LHS is 0 and the RHS is 1. This is indicative of how this inequality may fail for monotone increasing functions.

Proposition: If $f : [0,1] \to R$ is monotone increasing, then

$$\int_0^1 f'(x) \leq f(1) - f(0).$$

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