FOURIER ANALYSIS & METHODS 2020.03.02

JULIE ROWLETT

ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2019.03.04

There are several interesting facts about Bessel functions. Entire books have been written on these special functions.

1.1. Fun facts about Bessel functions.

Theorem 1 (Recurrence Formulas)**.** *For all x and ν*

$$(x^{-\nu}J_\nu(x))' = -x^{-\nu}J_{\nu+1}(x)$$
$$(x^{\nu}J_\nu(x))' = x^{\nu}J_{\nu-1}(x)$$
$$xJ_\nu'(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x)$$
$$xJ_\nu'(x) + \nu J_\nu(x) = xJ_{\nu-1}(x)$$
$$xJ_{\nu-1}(x) + xJ_{\nu+1}(x) = 2\nu J_\nu(x)$$
$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x)$$

Proof: Can you guess what we do? That's right - use the definition!!!! First,

$$x^{-\nu}J_\nu(x) = \sum_{n\geq 0} \frac{(-1)^n \frac{x^{2n}}{2^{2n+\nu}}}{n!\Gamma(n+\nu+1)}.$$

Take the derivative of the sum termwise. This is totally legitimate because this series converges locally uniformly in $\mathbb{C}$. So, we compute

$$\sum_{n\geq 1} \frac{(-1)^n 2n \frac{x^{2n-1}}{2^{2n+\nu}}}{n!\Gamma(n+\nu+1)} = \sum_{m\geq 0} \frac{(-1)^{m+1} 2(m+1) \frac{x^{2m+1}}{2^{2m+2+\nu}}}{(m+1)!\Gamma(m+2+\nu)}.$$

Above we re-indexed the sum by defining $n = m+1$. Next we do some simplifying around

$$= -\sum_{m\geq 0} \frac{(-1)^m \frac{x^{2m+1}}{2^{2m+1+\nu}}}{m!\Gamma(m+2+\nu)} = -x^{-\nu} \sum_{m\geq 0} \frac{(-1)^m \frac{x^{2m+1+\nu}}{2^{2m+1+\nu}}}{m!\Gamma(m+2+\nu)} = -x^{-\nu}J_{\nu+1}(x).$$

Next we compute similarly the derivative of $x^\nu J_\nu$ is

$$\sum_{n\geq 0} \frac{(-1)^n (2n+2\nu) \frac{x^{2n+2\nu-1}}{2^{2n+\nu}}}{n!\Gamma(n+\nu+1)}.$$

We factor out a 2 to get

$$\sum_{n\geq 0} \frac{(-1)^n (n+\nu) \frac{x^{2n+2\nu-1}}{2^{2n+\nu-1}}}{n!\Gamma(n+\nu+1)}.$$

Note that

$$\Gamma(n+\nu+1) = (n+\nu)\Gamma(n+\nu) \implies \frac{(n+\nu)}{\Gamma(n+\nu+1)} = \frac{1}{\Gamma(n+\nu)}.$$

So, above we have

$$\sum_{n\geq 0} \frac{(-1)^n \frac{x^{2n+2\nu-1}}{2^{2n+\nu-1}}}{n!\Gamma(n+\nu)} = x^\nu J_{\nu-1}(x).$$

To do the third one it is basically expanding out the first one:

$$(x^{-\nu} J_\nu(x))' = -\nu x^{-\nu-1} J_\nu + x^{-\nu} J_\nu' = -x^{-\nu} J_{\nu+1}.$$

Multiply through by $x^{\nu+1}$ to get

$$-\nu J_\nu + x J_\nu' = -x J_{\nu+1}.$$

We do similarly in the second formula:

$$\nu x^{\nu-1} J_\nu + x^\nu J_\nu' = x^\nu J_{\nu-1}.$$

Multiply by $x^{-\nu+1}$ to get

$$\nu J_\nu + x J_\nu' = x J_{\nu-1}.$$

Next, to get the fifth formula, subtract the third formula from the fourth. Finally, to get the sixth formula, add the third formula to the fourth.

We shall prove two lovely facts about the Bessel functions. The following fact is a theory item!

1.2. The generating function for the Bessel functions.

This is a lovely, follow your nose and use the definitions type of proof.

Theorem 2. *For all x and for all $z \neq 0$, the Bessel functions, J_n satisfy*

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}(z-\frac{1}{z})}.$$

Proof. We begin by writing out the familiar Taylor series expansion for the exponential functions

$$e^{xz/2} = \sum_{j\geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!},$$

and

$$e^{-x/(2z)} = \sum_{k\geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!}.$$

These converge beautifully, absolutely and uniformly for z in compact subsets of $\mathbb{C} \setminus \{0\}$. So, since we presume that $z \neq 0$, we can multiply these series and fool around with them to try to make the Bessel functions pop out... Thus, we write

bessel1
$$e^{xz/2}e^{-x/(2z)} = \sum_{j\geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!} \sum_{k\geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!} = \sum_{j,k\geq 0} (-1)^k \left(\frac{x}{2}\right)^{j+k} \frac{z^{j-k}}{j!k!}. \tag{1.1}$$

Here is where the one and only clever idea enters into this proof, but it's rather straightforward to come up with it. We would like a sum with $n = -\infty$ to ∞. So we look around into the above expression on the right, hunting for something which ranges from $-\infty$ to ∞. The only part which does this is $j - k$, because each of j and k range over 0 to ∞. Thus, we keep k as it is, and we let $n = j - k$. Then $j + k = n + 2k$, and $j = n + k$. However, now, we have $j! = (n + k)!$, but this is problematic if $n + k < 0$. There were no negative factorials in our original expression! So, to remedy this, we use the equivalent definition via the Gamma function,

$$j! = \Gamma(j + 1), \quad k! = \Gamma(k + 1).$$

Moreover, we observe that in (bessel1 1.1), $j!$ and $k!$ are for j and k non-negative. We also observe that

$$\frac{1}{\Gamma(m)} = 0, \quad m \in \mathbb{Z}, \quad m \leq 0.$$

Hence, we can write

$$e^{xz/2}e^{-x/(2z)} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{z^n}{\Gamma(n+k+1)k!}.$$

This is because for all the terms with $n + k + 1 \leq 0$, which would correspond to $(n+k)!$ with $n+k < 0$, those terms ought not to be there, but indeed, the $\frac{1}{\Gamma(n+k+1)}$ causes those terms to vanish!

Now, by definition,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k!\Gamma(k + n + 1)}.$$

Hence, we have indeed see that

$$e^{xz/2}e^{-x/(2z)} = \sum_{n=-\infty}^{\infty} J_n(x)z^n.$$

□

1.3. **Integral representation of the Bessel functions.** Let $z = e^{i\theta}$ for $\theta \in \mathbb{R}$. Then the theorem on the generating function for the Bessel functions says

$$\sum_{n\in\mathbb{Z}} J_n(x)z^n = e^{\frac{xz}{2} - \frac{x}{2z}}.$$

So, we use the fact that

$$\frac{1}{e^{i\theta}} = e^{-i\theta},$$

together with this formula to see that

$$\sum_{n\in\mathbb{Z}} J_n(x)e^{in\theta} = e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})}.$$

By Euler's formula,

$$\sum_{n\in\mathbb{Z}} J_n(x)e^{in\theta} = e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta).$$

Therefore, the left side is the Fourier expansion of the function on the right. OMG!!! Hence, the Bessel functions are actually *Fourier coefficients* of this function! So,

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta}e^{-in\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x\sin\theta - n\theta) + i\sin(x\sin\theta - n\theta)d\theta.$$

Note that

$$\sin(x\sin(-\theta) - n(-\theta)) = \sin(-x\sin\theta - n(-\theta)) = -\sin(x\sin\theta - n\theta).$$

So the sine part is odd and integrates to zero. We therefore have

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x\sin\theta - n\theta)d\theta.$$

This formula can be super useful. For example, we see that the Bessel functions have yet another property similar to their straight-laced Swedish ancestors, the sine and cosine. They satisfy $|J_n(\theta)| \leq 1 \forall x$.

1.4. **Applications to solving PDEs in circular type regions.** We shall now see how to generalize our Bessel function techniques to solve problems on pieces of circular sectors. Consider a circular sector of radius ρ and opening angle α. In the eyes of polar coordinates, this is a rectangle, $[0,\rho]\times[0,\alpha]$. That is, this set in $\mathbb{R}^2$ is in polar coordinates

$$\{(r,\theta)\in\mathbb{R}^2 : 0 \leq r \leq \rho, \text{ and } 0 \leq \theta \leq \alpha\}.$$

This is much the same as how we describe a rectangle using *rectangular* coordinates, (x,y).

To solve both the heat equation and the wave equation in a circular sector, we can use the same SLP and Fourier series style techniques we used on rectangles. The homogeneous heat equation is:

$$\partial_t u + \Delta u = 0, \quad \Delta = -\partial_{xx} - \partial_{yy}.$$

The homogeneous wave equation is:

$$u_{tt} + \Delta u = 0.$$

If we have neat and tidy (self-adjoint) boundary conditions, we can use separation of variables. Writing our function as $T(t)S(x,y)$, we obtain the equations:

heat $T'S + T\Delta S = 0$ which, dividing by the product TS becomes

$$\frac{\Delta S}{S} = -\frac{T'}{T} = \text{ constant.}$$

wave $T''S + T\Delta S = 0$ which, dividing by the product TS becomes

$$\frac{\Delta S}{S} = -\frac{T''}{T} = \text{ constant.}$$

So we see that in both cases we need to solve an equation of the form

$$\Delta S = \lambda S, \quad \lambda \text{ is a constant.}$$

After we solve this, we can then continue with solving both the heat equation and the wave equation.

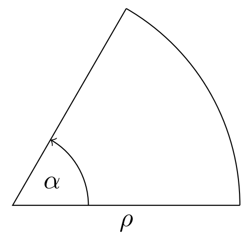

FIGURE 1. A circular sector of opening angle α and radius ρ.

1.5. **Dirichlet boundary condition on a circular sector.** Let's assume that we have the Dirichlet boundary condition on the boundary of the circular sector. So, we are looking for a function S which is zero on the boundary.

The boundary condition in polar coordinates is:

$$r = \rho, \quad \theta = 0, \quad \theta = \alpha.$$

So, it makes a lot more sense to use these coordinates. To proceed, we need to *write the operator using polar coordinates also!* We have previously computed in an exercise that in polar coordinates, the operator is:

$$\Delta = -\partial_{rr} - r^{-1}\partial_r - r^{-2}\partial_{\theta\theta}.$$

Let us try to solve $\Delta S = \lambda S$ in the circular sector using separation of variables. So, we have

$$R(r) \text{ and } \Theta(\theta).$$

The first one only depends on the r coordinate, whereas the second one only depends on the θ coordinate. Now, our PDE is:

$$-R''(r)\Theta(\theta) - r^{-1}R'(r)\Theta(\theta) - r^{-2}\Theta''(\theta)R(r) = \lambda R(r)\Theta(\theta).$$

First, we multiply everything by r^2, then we divide it all by ΘR to get

$$\frac{-r^2R'' - rR'}{R} - \frac{\Theta''}{\Theta} = \lambda \implies \frac{-r^2R'' - rR'}{R} - \lambda r^2 = \frac{\Theta''}{\Theta}.$$

Since the two sides depend on different variables, they are both constant. It turns out that the Θ side is much easier to deal with, so we look at solving it:

$$\frac{\Theta''}{\Theta} = \mu, \quad \Theta(0) = \Theta(\alpha) = 0.$$

We have solved such an equation a few times before. There are no non-zero solutions for $\mu > 0$. For $\mu < 0$ solutions are, up to constant factors,

$$\Theta_m(\theta) = \sin\left(\frac{m\pi\theta}{\alpha}\right), \quad \mu_m = -\frac{m^2\pi^2}{\alpha^2}.$$

As a consequence, we get the equation for R,

$$\frac{-r^2R'' - rR'}{R} - \lambda r^2 = \mu_m.$$

We multiply this equation by R, obtaining

$$-r^2R'' - rR' - \lambda r^2 R = \mu_m R.$$

This is equivalent to

$$r^2R'' + rR' + (\lambda r^2 + \mu_m)R = 0.$$

We make a small clever change of variables. Let

$$x = \sqrt{\lambda} r, \quad f(x) := R(r), \quad r = \frac{x}{\sqrt{\lambda}}.$$

Then by the chain rule

$$R'(r) = \sqrt{\lambda} f'(x), \quad R''(r) = \lambda f''(x).$$

So, the equation becomes

$$\left(\frac{x^2}{\lambda}\right) \lambda f''(x) + \frac{x}{\sqrt{\lambda}} \sqrt{\lambda} f'(x) + (x^2 + \mu_m) f(x) = 0.$$

This simplifies, recalling that $\mu_m = -m^2\pi^2/\alpha^2$,

besseleq

$$x^2 f''(x) + x f'(x) + (x^2 - m^2\pi^2/\alpha^2) f(x) = 0. \tag{1.2}$$

This is the definition of Bessel's equation of order $\frac{m\pi}{\alpha}$. Consequently, a solution to this equation is

$$J_{m\pi/\alpha}(x) = J_{m\pi/\alpha}(\sqrt{\lambda} r).$$

To satisfy the boundary condition, we would like

$$J_{m\pi/\alpha}(\sqrt{\lambda}\rho) = 0.$$

So, $\sqrt{\lambda}\rho$ should be a point at which this Bessel function vanishes. We have a useful fact about these zeros.

Theorem 3. *The Bessel function $J_{m\pi/\alpha}$ has infinitely many positive zeros which can be indexed as*

$$\{z_{m,k}\}_{k \geq 1},$$

where $z_{m,k}$ is the k^{th} positive zero.

Consequently, we shall have

$$J_{m\pi/\alpha}(z_{m,k} r/\rho), \quad \lambda_{m,k} = \frac{z_{m,k}^2}{\rho^2}.$$

We therefore have the collection of functions

$$S_{m,k}(\theta, r) = \sin(m\pi\theta/\alpha) J_{m\pi/\alpha}\left(\frac{z_{m,k} r}{\rho}\right).$$

Now we may obtain the time part of the solution.

heat Let us look for a solution to the homogeneous heat equation which satisfies

$$u(r, \theta, 0) = f(r, \theta).$$

Then, the partner functions T shall be given by:

$$\frac{\Delta S}{S} = -\frac{T'}{T} = \lambda_{m,k} \implies T_{m,k}(t) = A_{m,k} e^{-\lambda_{m,k} t}.$$

By superposition our full solution is therefore

$$u(r, \theta, t) = \sum_{m,k} A_{m,k} e^{-\lambda_{m,k} t} S_{m,k}(r, \theta).$$

wave Let us look for a solution to the homogeneous wave equation which satisfies

$$w(r,\theta,0)=g(r,\theta), \quad w_t(r,\theta,0)=0.$$

$$\frac{\Delta S}{S}=-\frac{T''}{T}=\lambda_{m,k} \implies T_{m,k}(t)=a_{m,k}\cos(z_{m,k}t/\rho)+b_{m,k}\sin(z_{m,k}t/\rho).$$

By superposition our full solution is therefore

$$w(r,\theta,t)=\sum_{m,k}(a_{m,k}\cos(z_{m,k}t/\rho)+b_{m,k}\sin(z_{m,k}t/\rho))S_{m,k}(r,\theta).$$

To determine the coefficients, we shall use the following theorem.

Theorem 4. *The set of functions*

$$\sin(m\pi\theta/\alpha)J_{m\pi/\alpha}\left(\frac{z_{m,k}r}{\rho}\right), \quad k\geq 0, \quad m\geq 1$$

are an orthogonal basis for $\mathcal{L}^2$ on the sector of radius ρ and opening angle α. Above, $z_{m,k}$ is the k^{th} positive zero of $J_{m\pi/\alpha}$.

Consequently, for the heat equation we demand

$$u(r,\theta,0)=\sum_{m,k}A_{m,k}S_{m,k}(r,\theta)=f(r,\theta),$$

which shows us that the coefficients should be

$$A_{m,k}=\frac{\langle f,S_{m,k}\rangle}{||S_{m,k}||^2},$$

where

$$\langle f,S_{m,k}\rangle=\int_0^\alpha\int_0^\rho f(r,\theta)\overline{S_{m,k}(r,\theta)}rdrd\theta,$$

and

$$||S_{m,k}||^2=\int_0^\alpha\int_0^\rho |S_{m,k}(r,\theta)|^2rdrd\theta.$$

For the wave equation we demand

$$w(r,\theta,0)=\sum_{m,k}a_{m,k}S_{m,k}(r,\theta)=g(r,\theta) \implies a_{m,k}=\frac{\langle g,S_{m,k}\rangle}{||S_{m,k}||^2}.$$

The second condition tells us what the other coefficients should be:

$$w_t(r,\theta,0)=\sum_{m,k}z_{m,k}/\rho b_{m,k}S_{m,k}(r,\theta)=0 \implies b_{m,k}=0\forall m,k.$$

1.6. **Bessel functions for Neumann boundary condition.** This theorem is another type of "adult spectral theorem."

Theorem 5. *Assume that $\nu\geq 0$ and $\rho>0$. Assume that $c\geq -\nu$. Let*

$$\{z_k\}_{k\geq 1}$$

be the positive zeros of $cJ_\nu(x)+xJ'_\nu(x)$, and let

$$\psi_k(x)=J_\nu(z_kx/\rho).$$

If $c>-\nu$ then $\{\psi_k\}_{k\geq 1}$ is an orthogonal basis for $\mathcal{L}^2_w$ on the interval $(0,b)$ for the weight function $w(x)=x$. If $c>-\nu$, then $\{\psi_k\}_{k\geq 0}$ is an orthogonal basis for $\mathcal{L}^2_w$ on the interval $(0,b)$ for the weight function $w(x)=x$, with $\psi_0(x)=x^\nu$.

Let us see how to apply this theorem when we are solving the heat (and wave) equations with the Neumann boundary condition. We follow the same procedure as for the heat equation. Let us name the sector

$$\Sigma.$$

$$u_t + \Delta u = 0, \quad \text{inside } \Sigma,$$
$$u(r, \theta, 0) = v(r, \theta) \text{ inside } \Sigma$$

the outward pointing normal derivative of $u = 0$ on the boundary of Σ.

We do the same procedure as before. We arrive at the equation for the Θ part:

$$\Theta'' = \mu\Theta, \quad \Theta'(0) = \Theta'(\alpha) = 0.$$

You can do the exercise to show that the only solutions are for $\mu < 0$, and to satisfy the boundary conditions, up to constant multiples

$$\Theta_m(\theta) = \cos(m\pi/\alpha), \quad \mu_m = -\frac{m^2\pi^2}{\alpha^2}, \quad m \geq 0.$$

Then, we again arrive at the Bessel equation of order $m\pi/\alpha$ for the function R. So, we get that

$$R_m(r) = J_{\nu_m}(\sqrt{\lambda}r), \quad \nu_m = m\pi/\alpha.$$

The boundary condition for R_m is that

$$R_m'(\rho) = 0.$$

So, this means we need

$$\sqrt{\lambda}J_{\nu_m}'(\sqrt{\lambda}\rho) = 0.$$

In other words, $\sqrt{\lambda}$ needs to be a solution of the equation

$$xJ_{\nu_m}'(\rho x) = 0.$$

If z_k is a solution to

$$xJ_{\nu_m}'(x) = 0,$$

then

$$z_k J_{\nu_m}'(z_k) = 0 \implies \frac{z_k}{\rho} J_{\nu_m}'(z_k\rho/\rho) = 0.$$

So, to satisfy the boundary condition, we need

$$\sqrt{\lambda} = \frac{z_k}{\rho} \implies \sqrt{\lambda}J_{\nu_m}'(\sqrt{\lambda}\rho) = 0.$$

Really, z_k also depends on m, so that is why we write $z_{m,k}$ to mean the k^{th} positive solution of the equation

$$xJ_{\nu_m}'(x) = 0.$$

Our function

$$R_{m,k}(r) = J_{\nu_m}(z_{m,k}r/\rho).$$

This also shows that

$$\lambda_{m,k} = \frac{z_{m,k}^2}{\rho^2}.$$

Now, we recall the equation for the partner function, T,

$$T_{m,k}'(t) = -\lambda_{m,k}T_{m,k}(t).$$

So, up to constant factors,

$$T_{m,k}(t) = e^{-\lambda_{m,k}t}.$$

To apply the theorem, we note that

$$\nu_m = m\pi/\alpha > 0 \forall m \in \mathbb{N}.$$

Therefore taking $c = 0$ in the theorem, $c \geq -\nu_m$ for all m. The theorem then tells us that the set

$$\{R_{m,k}(r)\}_{k\geq 1} = \{J_{\nu_m}(z_{m,k}r/\rho)\}_{k\geq 1}$$

is an orthogonal basis for $\mathcal{L}^2(0,\rho)$ with respect to integrating against rdr. We also know that the $\Theta_m(\theta)$ functions are an orthogonal basis for $\mathcal{L}^2(0,\alpha)$ with respect to integrating against $d\theta$. Consequently, the entire collection

$$S_{m,k}(r,\theta) = \Theta_m(\theta)R_{m,k}(r)$$

is an orthogonal basis for $\mathcal{L}^2(\Sigma)$. This is because integrating on $\mathcal{L}^2(\Sigma)$ in polar coordinates is integrating

$$\int_\Sigma v(r,\theta) r dr d\theta = \int_0^\rho \int_0^\alpha v(r,\theta) r dr d\theta.$$

So, the theorem says that we can expand the initial data in a Fourier series with respect to the orthogonal basis functions $S_{m,k}$. We therefore write the solution

$$u(r,\theta,t) = \sum_{m,k} \widehat{v_{m,k}} T_{m,k}(t) S_{m,k}(r,\theta),$$

where

$$\widehat{v_{m,k}} = \frac{\int_\Sigma v(r,\theta) S_{m,k}(r) r dr d\theta}{||S_{m,k}||^2}$$

$$= \frac{\int_0^r \int_0^\theta \sin(m\pi\theta/\alpha) J_{m\pi/\alpha}(z_{m,k}r/\rho) v(r,\theta) r dr d\theta}{\int_0^r \int_0^\theta \sin(m\pi\theta/\alpha)^2 J_{m\pi/\alpha}(z_{m,k}r/\rho)^2 r dr d\theta}.$$

1.7. **Exercises to be demonstrated.**

(1) Eö 28
(2) (5.5.2) A circular cylinder of radius ρ is at the constant temperature A. At time $t = 0$ it is tightly wrapped in a sheath of the same material of thickness δ, thus forming a cylinder of radius $\rho + \delta$. The sheath is initially at temperature B, and its outside surface is maintained at temperature B. If the ends of the new, enlarged cylinder are insulated, find the temperature inside at subsequent times.
(3) Eö 30
(4) Eö 52
(5) Eö 53
(6) (5.5.4) A cylindrical uranium rod of radius 1 generates heat within itself at a constant rate a (think radioactive material). Its ends are insulated and its circular surface is immersed in a cooling bath at temperature zero. Thus

$$u_t = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + a, \quad u(1,t) = 0.$$

First find the steady state temperature $v(r)$ in the rod. Then find the temperature in the rod if its initial temperature is zero.

1.8. **Exercises to be done oneself.**

(1) (5.2.4) Demonstrate the identity:

$$\int_0^x sJ_0(s)ds = xJ_1(x), \quad \int_0^x J_1(s)ds = 1 - J_0(x).$$

(2) (5.5.1) A cylinder of radius b is initially at the constant temperature A. Find the temperatures in it at subsequent times if its ends are insulated and its circular surface obeys Newton's law of cooling, $u_r + cu = 0$, $(c > 0)$.

(3) (5.5.5) Solve the problem

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} = 0 \text{ in } D = \{(r, \theta, z) : 0 \le r \le b, 0 \le z \le l\}$$
$$u(r, \theta, 0) = 0, \quad u(r, \theta, l) = g(r, \theta), \quad u(b, \theta, z) = 0.$$

(4) (5.5.6) Find the steady-state temperature in the cylinder $0 \le r \le 1$, $0 \le z \le 1$ when the circular surface is insulated, the bottom is kept at temperature 0, and the top is kept at temperature $f(r)$.

(5) Eö 29

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).