Filters and Signals, Filter Banks

A discrete signal is a (double) sequence

\[ X = \{ x_k \}_{k=-\infty}^{\infty} = (\ldots, x_{-1}, x_0, x_1, \ldots), \quad x_k \in \mathbb{R} \text{ (or } \mathbb{C}) \]

So \( X \) is a function \( X : \mathbb{Z} \to \mathbb{R} \) (or \( \mathbb{C} \)).

**Def:** \( x \in l^2 \iff \sum_{k=-\infty}^{\infty} |x_k|^2 < \infty \) (finite energy)

**Def:** A filter is an operator, \( H : X \to Y \), i.e.

\[ Y = HX \text{ is another signal} \]

\[ \begin{array}{c}
X \\
\downarrow \\
H \\
\uparrow \\
Y
\end{array} \]

**Def:** \( H \) is linear if \( H(\alpha x + \beta y) = \alpha H(x) + \beta H(y) \)

\( H \) is time invariant if \( H(Dx) = D H(x) \), \( \forall x \).

(note \( Dx \) is defined by \( (Dx)_k = x_{k-1} \))

\[ \def \delta = \{ \delta_k \}_{k=-\infty}^{\infty} \]

\[ \delta_k = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{else} \end{cases} \]

\[ \def \h = H \delta \text{ is called the impulse response of the filter } H. \]

Suppose now that a filter \( H \) is linear, time invariant (LTI) we then have

\[ D^n h = H(D^n \delta). \]

Take \( X = \{ x_k \}_{k=-\infty}^{\infty} \) arbitrary. Then

\[ X = \ldots + x_{-2} D^{-2} \delta + x_{-1} D^{-1} \delta + x_0 \delta + x_1 D \delta + x_2 D^2 \delta + \ldots \]

\[ \delta = \{ 0,0,\ldots,0,0,1,0,0,\ldots,0,0 \} \Rightarrow \]

\[ X_0 \delta = \{ x_0,\ldots,0,0,0,0,\ldots,0,0 \} \]

\[ D^{-1} \delta = \{ 0,0,\ldots,0,1,0,0,\ldots,0,0 \} \Rightarrow \]

\[ x_0 D^{-1} \delta = \{ x_0,0,\ldots,0,0,0,0,\ldots,0,0 \} \]

\[ D \delta = \{ 0,0,\ldots,0,0,1,0,0,\ldots,0,0 \} \Rightarrow \]

\[ x_0 D \delta = \{ x_0,0,\ldots,0,0,0,0,\ldots,0,0 \} \]

\[ \Rightarrow \]
\[ X = \ldots + 10, \ldots, 0, x_n, 0, 0, \ldots, 0, 0 \]
\[ \ldots + 0, \ldots, 0, 0, x_0, \ldots, 0, 0, 0 \]
\[ \ldots + 0, \ldots, 0, 0, 0, x_1, 0, \ldots, 0, 0 \]
\[ \ldots = \sum_{n=-\infty}^{\infty} x_n D^n \delta \]

Hence, if \( y = Hx \), we get

\[ y = Hx = H \left( \sum_{n=-\infty}^{\infty} x_n D^n \delta \right) = \sum_{n=-\infty}^{\infty} x_n D^n h \]

\[ = \sum_{n=-\infty}^{\infty} x_n h_{-n} \]

i.e.

\[ y = \sum_{k=-\infty}^{\infty} y_k \delta_{-k} \]

\[ y_k = \sum_{n=-\infty}^{\infty} x_n h_{k-n} = x_k h \]

\[ = h \ast x \]

Definition of discrete convolution

So LT1 filter is uniquely determined by its impulse response.

By a Finite Impulse Response filter (FIR)

One says Infinite Impulse Response (IIR) filter.

**Def:** An LT1 filter is causal if

\[ h_k = 0 \quad \text{for} \quad k < 0. \]

**Def (autocorrelation)**

\[ x \ast y = \sum_{k=-\infty}^{\infty} x_k y_{n-k} \]

\[ \text{Example:} \quad \text{Let} \quad h_k = \begin{cases} 1/2 & \text{where} \quad k = 0, 1 \\ 0 & \text{else} \end{cases} \]

Then

\[ g_k = \frac{1}{2} (x_k + x_{k-1}) \]

\[ \delta = \sum_{n=-\infty}^{\infty} x_n \delta_{n-k} \]

\[ \int_{-2}^{1} \]

This is a LT1 filter, that is also causal.
Example: Down sampling by 2, (denoted as $\downarrow 2$).

If $X = \{X_k\}_{k=-\infty}^{\infty}$

\[ (\downarrow 2) X = (\ldots, X_{-2}, 0, X_0, x_2, X_4, 0, \ldots) \]

Upsampling by 2, denoted by $\uparrow 2$.

\[ (\uparrow 2) X = (\ldots, X_{-2}, 0, X_{-1}, 0, X_0, 0, X_1, 0, X_2, 0, \ldots) \]

These are linear but not time invariant (why?)

**Def.** The $Z$-transform of a signal $X = \{X_k\}_{k=-\infty}^{\infty}$ is

\[ X(z) = \sum_{k=-\infty}^{\infty} X_k z^{-k}, \quad z \in \mathbb{C}. \]

We write $X \in \mathbb{Z}(z)$.

Here the argument must be written out to avoid confusion with DFT, $X(\omega) = \ldots$

Example: $X_k = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$.

Then $X \in \mathbb{Z}(z)$ with

\[ X(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}}, \quad (z \in \mathbb{C}, |z| < 1). \]

**The convolution theorem**

If $y = h \ast x$, then $Y(z) = H(z)X(z)$

and vice versa.

\[ Y(z) = \sum_{k=-\infty}^{\infty} z^{-k} \left( \sum_{h=-\infty}^{\infty} h_{k-h} X_h \right) \]

**Proof:**

\[ Y(z) = \sum_{k=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} z^{-k} h_{k-h} x_h = \sum_{h=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} z^{-k} h_{k-h} \uparrow (m = k-h) \]

\[ = \sum_{m=-\infty}^{\infty} z^{-m} X_m \uparrow (m) = H(z) \ast x(z), \quad \square \]

**The Discrete Fourier Transform (of infinite sequences)**

\[ X(\omega) = \sum_{k=-\infty}^{\infty} X_k e^{-j\omega k} \]

Note! In this def.

\[ e^{-j\omega k} \big|_{\omega = -2\pi / \omega} = e^{j\omega k} \big|_{\omega = \omega} \]

Other common notation: $\hat{X}(\omega) = \mathbb{X}(\omega)$.

Note that $\mathbb{X}(\omega) = \mathbb{X}(e^{j\omega}) = \left\{ z = e^{j\omega} \right\} \mathbb{X}(z)$

**Lemma** $\mathbb{X}(\omega)$ is $2\pi$-periodic, and

\[ x_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{X}(\omega)e^{j\omega k} \, d\omega. \]
Def: $H(w)$ is called the Frequency response.

Example: Let $x_k = e^{i\omega k}$, $|\omega| \leq \pi$.

Then $y_k = Hx_k$ is given by

$$y_k = \sum_{n=-\infty}^{\infty} h_n x_{k-n} = \sum_{n=-\infty}^{\infty} h_n e^{-i\omega(k-n)} = e^{i\omega k} \sum_{n=-\infty}^{\infty} h_n e^{-i\omega n} = e^{i\omega k} H(w).$$

Write $H(w) = |H(w)| e^{i\phi(w)}$.

Definition: $H$ has Linear Phase if $w \rightarrow \phi(w)$ is linear.
- $H$ is symmetric if $h_k = h_{-k}$.
- $H$ is anti-symmetric if $h_k = -h_{-k}$.
- The group delay $\tau$ of $H$ is: $\tau(w) = -\frac{d\phi}{dw}$.

Example: Let $h_0 = h_1 = \frac{1}{2}$, $h_k = 0$, $k \neq 0, 1$.

$\Rightarrow H(w) = \sum_{n=-\infty}^{\infty} h_n e^{-i\omega n} = \frac{1}{2} (1 + e^{-i\omega}) = e^{-i\omega/2} g(w) = \frac{1}{2} \left(1 - e^{-i\omega} \right)$.

b) $g_0 = \frac{1}{2}$, $g_1 = -\frac{1}{2}$, $g_k = 0$, $k \neq 0, 1$.

$\Rightarrow g(w) = \frac{1}{2} \left(1 - e^{-i\omega} \right)$.

Then

$$H(w) = \frac{1}{2} + 1 \cdot e^{-i\omega} + \frac{1}{2} \cdot e^{-2i\omega} = e^{-i\omega} \left(1 + \frac{1}{2} e^{i\omega} + \frac{1}{2} e^{-i\omega} \right) = e^{-i\omega} \frac{(1 + \cos(\omega))}{|H(w)|} \quad \phi(w) = -\omega \text{ linear,}$$

$\Rightarrow H$ has linear phase.
Example

If \( H \) is a LTI filter and \( x \) is a pure frequency: \( x_k = e^{i\omega k} \),
then

\[
y_k = (h \ast x)_k = \sum_{n=-\infty}^{\infty} h_n x_{k-n} = \sum_{n=-\infty}^{\infty} h_n e^{i\omega(k-n)}
\]

\[
= e^{i\omega k} \sum_{n=-\infty}^{\infty} h_n e^{-i\omega n} = H(\omega) e^{i\omega k}.
\]

If \( x_k = e^{i\omega_1 k} + e^{i\omega_2 k} \) (sum of two pure frequencies),
then

\[
y_k = H(\omega_1) e^{i\omega_1 k} + H(\omega_2) e^{i\omega_2 k}
\]

\[
= |H(\omega_1)| e^{i\phi(\omega_1)} e^{i\omega_1 k} + |H(\omega_2)| e^{i\phi(\omega_2)} e^{i\omega_2 k}
\]

\[
= |H(\omega_1)| e^{i\omega_1 (k-1)} + |H(\omega_2)| e^{i\omega_2 (k-1)}.
\]

Note the shift \( k \rightarrow k-1 \).

Filter banks

Def. A sequence of signals \( \{\varphi^{(n)}\}_{n=-\infty}^{\infty} \)

\[
(\ldots, \varphi^{(n)}, \varphi_0^{(n)}, \varphi_1^{(n)}, \varphi_2^{(n)}, \ldots)
\]

is called a basis if every \( x \) can be written as

\[
x = \sum_{n} c_n \varphi^{(n)}.
\]

Here we need to restrict \( x \)

Def. \( \|x\| = \sqrt{\sum |x_n|^2} \) (this is sometimes denoted by \( \|x\|_2 \) or \( \|x\|_{L_2} \)).

We say that \( x^{(n)} \rightarrow x \) in norm if

\[
\|x^{(n)} - x\| \rightarrow 0, \quad \text{when } n \rightarrow \infty.
\]

Def. A Hilbert space is a "complete inner product space".

Ex. \( L_2 \), \( \langle x, y \rangle = \sum x_k \overline{y}_k \) (note complex conjugate on y)

\[
\Rightarrow \|x\| = \sqrt{\langle x, x \rangle}.
\]
Cauchy-Schwarz: \[ | \langle x, y \rangle | \leq \| x \| \| y \| \]

A Cauchy sequence is a sequence \( \{ x^{(n)} \}_{n=1}^{\infty} \) such that \( \forall \varepsilon > 0 \) \( \exists N \) such that
\[ \| x^{(n)} - x^{(m)} \| < \varepsilon \quad \text{for} \quad n, m > N. \]

**Definition:** A space is complete if every Cauchy sequence is convergent.

- An orthogonal basis \( \{ \psi^{(n)} \} \) is a basis that satisfies
\[ \langle \psi^{(i)}, \psi^{(j)} \rangle = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{else} \end{cases} \]

We can write
\[ x = \sum_{n} c_{n} \psi^{(n)} \quad \text{where} \quad c_{n} = \langle x, \psi^{(n)} \rangle. \]

(Compare with the familiar concepts in \( \mathbb{R} \))

**Definition:** The Haar basis consists of two families of functions.

\[ \psi_{k} = \begin{cases} \frac{1}{\sqrt{2}} & \text{when} \quad k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \quad \psi^{(k)} = \begin{cases} \frac{1}{\sqrt{2}} & \text{for} \quad k = 0 \\ -\frac{1}{\sqrt{2}} & \text{for} \quad k = 1 \\ 0 & \text{otherwise} \end{cases} \]

Let maps \( (\varphi^{(2n)})_{k} = \psi_{k-2n} \), \( (\varphi^{(2n+1)})_{k} = \psi_{k-2n} \).

The coordinates of a sequence \( x = (x_{k})_{k=-\infty}^{\infty} \) in this basis are

\[ y_{n}^{(0)} := \sum_{k=2n}^{2(n+1)} \frac{1}{\sqrt{2}} (x_{2n} + x_{2n+1}); \quad (\text{mean}) \]
\[ y_{n}^{(1)} := \sum_{k=2n+1}^{2(n+1)} \frac{1}{\sqrt{2}} (x_{2n} - x_{2n+1}); \quad (\text{difference}) \]
Recall the up- and down-samplings:

\( (V^2)_X = (\ldots, X_{2}, X_{1}, X_{0}, 0, 0, \ldots) \)

\( (U^2)_X = (\ldots, X_{2}, X_{1}, X_{0}, 0, 0, \ldots) \)

We can obtain the coordinates \( y_{n}^{(0)} \) and \( y_{n}^{(1)} \) using the two filters:

\[
\begin{align*}
H^* \rightarrow y^{(0)} & \quad \text{and} \\
G^* \rightarrow y^{(1)} & \quad \text{for} \quad n = \ldots, 2, 1, 0, 0, \ldots
\end{align*}
\]

\[
\text{this procedure is called } \mathbf{analysis}\]

\[
\text{The } \mathbf{synthesis} \text{ is carried out similarly}
\]

\[
\begin{align*}
X_k &= \sum_n y_{n}^{(1)} h_{k-2n} + \sum_n y_{n}^{(2)} h_{k-2n} = \sum_n y_{n}^{(0)} h_{k-2n} + \sum_n y_{n}^{(0)} h_{k-2n} \\
&= (V^{(0)} \ast h)_k + (V^{(0)} \ast g)_k, \\
&= (\uparrow 2 y^{(0)})_k, \quad V = (\uparrow 2 y^{(0)})_k
\end{align*}
\]

\[
\text{Note that:}
\sum_n y_{n} v_{k-2n} = \sum_n v_{2n} \tilde{y}_{k-2n} = \sum_n \tilde{y}_{k-n} = (V^* h)_k.
\]

\[
\text{Therefore,}
X = H(\uparrow 2 y^{(1)}) + G(\uparrow 2 y^{(0)})
\]

\[
\text{where all odd, } v_{2n+1} = 0.
\]
\[ X = \sum \langle x, \varphi^{(2n)} \rangle \varphi^{(2n)} + \sum \langle x, \varphi^{(2n+1)} \rangle \varphi^{(2n+1)} \]

projection onto even

projection onto odd

DFT and the \( Z \)-transform of \( \Omega_z \) and \( \Omega_w \)

Recall \( y = \Omega_z x = (\ldots, x_{-1}, x_0, x_1, \ldots) \Rightarrow y_k = x_{2k} \)

\[ Y(z) = \sum_{k=-\infty}^{\infty} y_k z^{-k} = \sum_{k=-\infty}^{\infty} x_{2k} z^{-k} = \sum_{k=-\infty}^{\infty} x_k (z^2)^{-k} \]

\[ = \frac{1}{2} \left[ \sum_{k=-\infty}^{\infty} x_{2k} (z^2)^{-k} + \sum_{k=-\infty}^{\infty} x_{2k+1} (z^2)^{-k} \right] 
+ \frac{1}{2} \left[ \sum_{k=-\infty}^{\infty} x_{2k} (z^2)^{-k} - \sum_{k=-\infty}^{\infty} x_{2k+1} (z^2)^{-k} \right] \]

\[ = \frac{1}{2} \left( X(z^2) + X(-z^2) \right). \]

With \( z = e^{i\omega} \), \( z^2 = e^{i\omega/2} \), \( z^{-2} = e^{i(\omega/2 + \pi)} \)

we get the DFT:

\[ Y(w) = \frac{1}{2} \left( X(w) + X(w + \pi) \right). \]

The term \( X(w + \pi) \) is an "alias component" which can be removed with a "perfect filter" \( H, G \).

Upsampling

\[ y = \Omega_{2w} x = (\ldots, x_{-1}, x_0, x_1, 0, 0, x_1, 0, \ldots) \]

\[ (\ldots y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots) \]

Then \( Y(z) = \sum_{k=-\infty}^{\infty} y_k z^{-k} \)

\[ = \sum_{k=-\infty}^{\infty} y_{2k} z^{-2k} = \sum_{k=-\infty}^{\infty} x_k z^{-2k} = \overline{X(z^2)} \]

and \( Y(w) = X(2w) \).

It is clear that:

\[ \Omega_{2w} \Omega_{2w} X = X \]

in general.

Let now \( u = \Omega_{2w} \Omega_{2w} X \), then

\[ U(z) = \sum_{k=-\infty}^{\infty} u_k z^{-k} = \sum_{k=-\infty}^{\infty} x_{2k} z^{-2k} = \frac{1}{2} \left[ \sum_{k=-\infty}^{\infty} x_{2k} z^{-2k} + \sum_{k=-\infty}^{\infty} x_{2k+1} z^{-2k} \right] 
+ \frac{1}{2} \left[ \sum_{k=-\infty}^{\infty} x_{2k} z^{-2k} - \sum_{k=-\infty}^{\infty} x_{2k+1} z^{-2k} \right] \]

\[ = \frac{1}{2} \left( X(z^2) + X(z^2) \right) \]

\[ U(w) = \frac{1}{2} \left( X(w) + X(w + \pi) \right). \]
Perfect resolution

A Filter Bank.

In the analysis part, the signal is split into a high frequency part and a low frequency part, which gives the signals $y^{(0)}$ and $y^{(1)}$. These then pass the synthesis part to make a reconstruction of the signal. If $\hat{X} = X$, we have achieved "perfect resolution".

Expressed in terms of Z-transform, we have for the high frequency part

$$X^{(0)}(z) = \frac{1}{2} \left[ H(z) \tilde{H}^*(z) \tilde{X}^*(z) + \tilde{H}^*(z) \tilde{X}^*(z) \right]$$

Similarly

$$X^{(1)}(z) = \frac{1}{2} \left[ G(z) \tilde{G}^*(z) \tilde{X}^*(z) + \tilde{G}^*(z) \tilde{X}^*(z) \right]$$

Which gives

$$\hat{X}(z) = \frac{1}{2} \left[ H(z) \tilde{H}^*(z) + G(z) \tilde{G}^*(z) \right] X(z) + \frac{1}{2} \left[ H(z) \tilde{H}^*(z) + G(z) \tilde{G}^*(z) \right] X(z)$$

We therefore achieve perfect resolution if and only if

$$H(z) \tilde{H}^*(z) + G(z) \tilde{G}^*(z) = 2 \quad H(z) \tilde{H}^*(z) + G(z) \tilde{G}^*(z) = 0$$

How can one construct such filters? The "product filters" are obtained by the ansatz

$$G(z) = -z^{-L} \tilde{H}^*(z) \quad L \text{ odd}$$

$$\tilde{G}(z) = -z^{-L} \tilde{H}^*(z)$$
Recall that the *-marked filters are time reversed.

\[
\tilde{g}_k(2) = \sum_{k=-\infty}^{\infty} g^*_k z^k = \sum_{k=-\infty}^{\infty} g^*_k (\frac{1}{z})^{-k} = G^*(z)
\]

and therefore

\[
\tilde{g}^*_k(2) = \tilde{G}^*_k(\frac{1}{2}) = -\left(\frac{1}{2}\right)^{-k} H^*_k(\frac{1}{2})
\]

\[
\tilde{H}^*_k(2) = \tilde{H}(\frac{1}{2})^k
\]

Therefore

\[
H(z) \tilde{h}^*(z) + G(z) \tilde{g}^*(z) = H(z) H^*(z) + \tilde{H}(\frac{1}{2}) \tilde{H}(-\frac{1}{2})^{-L} \tilde{H}(-\frac{1}{2}) = 0
\]

and

\[
\tilde{H}^*(z) \tilde{g}^*(z) + \tilde{G}^*(z) \tilde{h}^*(z) = \tilde{H}(\frac{1}{2}) \tilde{H}^*(\frac{1}{2})^{-L} \tilde{H}(-\frac{1}{2}) \tilde{H}(-\frac{1}{2})
\]

The condition for perfect reconstruction is now given in terms of \( h \) and \( \tilde{h} \):

\[
H(z) \tilde{h}^*(z) + H^*(z) \tilde{h}(-z) = 0
\]

and a product filter is defined by

\[
P(z) = H(z) H^*(z).
\]

Then, the perfect reconstruction condition becomes

\[
P(z) + P(-z) = 2, \quad \text{i.e.}
\]

\[
2P_0 + z^{-1} \sum_{k=-\infty}^{\infty} P_{2k} z^{-2k} = 2.
\]

It follows that \( P_0 = 1 \), \( P_{2n} = 0 \) (\( n \neq 0 \)), and \( P_{2n+1} \) can be chosen arbitrarily.

Given \( P(z) \), we can choose \( H(z) \) and \( \tilde{H}(z) \), but this may be done in many different ways.
Orthogonal Filter Banks

Filters that satisfy $H(z) = \tilde{H}(z)$ and $\tilde{G}(z) = \tilde{G}(z)$ are called orthogonal. Then

$$P(z) = H(z) \tilde{H}^*(z) = H(z) \tilde{H}(1/\bar{z}) = H(z) \bar{H}(1/z).$$

In the DFT version

$$p(w) = H(w) \bar{H}(w) = |H(w)|^2 > 0.$$  

Then $p(w)$ is even and real-valued which implies that the coefficients are symmetric, and perfect reconstruction follows from

$$|H(w)|^2 + |H(w+m)|^2 = 2.$$  

The reason for "writing orthogonal" (Why $H(z) = \tilde{H}(z)$ means orthogonal?)

Recall that in the example of the Haar basis, we had

$$\psi_k = \begin{cases} \frac{1}{\sqrt{2}}, & k = 0 \\ 0, & \text{else} \end{cases}, \quad \psi_k = \begin{cases} \frac{1}{\sqrt{2}}, & k = 1 \\ 0, & \text{else} \end{cases}$$

and we constructed the basis functions

$$\psi_{2n} = \psi_{k-2n}, \quad \psi_{2n+1} = \psi_{k-2n+1}.$$  

and finally at the end

$$y_n^{(1)} = \langle x, \varphi^{(2n)} \rangle, \quad y_n^{(2)} = \langle x, \varphi^{(2n+1)} \rangle$$

$$x = \sum_n \langle x, \varphi^{(2n)} \rangle \psi^{(2n)} + \sum_n \langle x, \psi^{(2n+1)} \rangle \psi^{(2n+1)}.$$  

Biorthogonal basis

Often it is advantageous to work with "biorthogonal" bases, which corresponds to expressions of the form

$$x = \sum_n \langle x, \varphi^{(2n)} \rangle \psi^{(2n)} + \sum_n \langle x, \psi^{(2n+1)} \rangle \varphi^{(2n+1)}.$$  

The basis functions must then satisfy

$$\langle \varphi^{(2k)}, \varphi^{(2n)} \rangle = \delta_{k,n},$$  

$$\langle \psi^{(2k+1)}, \varphi^{(2n+1)} \rangle = \delta_{k,n},$$  

$$\langle \varphi^{(2k)}, \psi^{(2n+1)} \rangle = \langle \varphi^{(2k+1)}, \varphi^{(2n)} \rangle = 0.$$  

We shall return to these relations when discussing multi-resolution analysis.
Design of filter banks

This can be done in three steps:

Step 1. Find $p(z)$ that satisfies the condition for perfect reconstruction.

Step 2. Factorize $p(z)$:

$$p(z) = H(z) \hat{H}^*(z).$$

Step 3. Define the high pass filters by

$$
\begin{cases}
G(z) = - \frac{1}{2} H(-z)
\\
\hat{G}(z) = - \frac{1}{2} \hat{H}(-z)
\end{cases}
$$

where $L$ is odd.

**Example**

The Haar basis

$$p(z) = \frac{1}{\sqrt{2}} \left( z + z^{-1} \right)$$

$$= \frac{1}{\sqrt{2}} \left( z + 1 \right) \cdot \frac{1}{\sqrt{2}} \left( \frac{1}{z} + 1 \right) = \frac{1}{\sqrt{2}} H(z) \frac{1}{\sqrt{2}} \hat{H}^*(z).$$