1 General properties of ω -limit sets. La Salle's invariance principle and it's applications to asymptotic stability.§5.2.

Example. An elementary introduction to LaSalle's invariance principle.

We like to investigate stability of equilibrium point in the origin for the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - x_2^3 \end{aligned}$$

Using the simple test function $V(x_1, x_2) = x_1^2 + x_2^2$ we observe that it is a Lyapunov function for the system:

$$V_f(x_1, x_2) = \nabla V \cdot f(x_1, x_2) = 2x_1x_2 - 2x_1x_2 - 2x_2^4 = -2x_2^4 \leq 0$$

and the origin is a stable equilibrium point. But V is not a strong Lyapunov function, because $V_f(x_1, x_2) = 0$ not only in the origin, but on the whole x_1 - axis where x_2 is zero.

On the other hand considering the vector field of velocities of this system on the x_1 - axis, we observe that they are crossing the x_1 - axis (even are orthogonal to it in this particular example) in all points except the origin. It means that all trajectories of the system cross and immediately leave the x_1 - axis that is the line where $V_f(x_1, x_2) = 0$ (the Lyapunov function is not strong). This observation shows that in fact the Lyapunov function $V(\varphi(t, \xi))$ is strictly monotone decreasing along trajectories $\varphi(t, \xi)$ everywhere except discret time moments, when $\varphi(t, \xi)$ crosses the x_1 - axis.

More explicitely we can express the same effect in polar coordinates r and θ:

$$\left(r^2\right)' = -2r^4 \sin^4 \theta$$

We can therefore conclude that $V(\varphi(t,\xi)) \searrow 0$ as $t \to \infty$ and therefore, the origin is asymptotically stable equilibrium of this system of equations.

One can also get a more explicit picture of this dynamics by looking on the equation for the polar angle θ:

$$
\begin{aligned}
\left(\frac{x_2}{x_1}\right)' &= (\tan(\theta))' = \frac{\theta'}{\cos^2(\theta)} \\
\frac{x_2' x_1 - x_1' x_2}{x_1^2} &= \frac{(-x_1 - x_2^3)\, x_1 - (x_2)\, x_2}{x_1^2} \\
&= \frac{(-x_1^2 - x_2^2 - x_1 x_2^3)}{x_1^2} = \frac{-r^2 - \cos\theta \sin^3\theta\, r^4}{r^2 \cos^2\theta}
\end{aligned}
$$

$$
\begin{aligned}
\theta' &= -1 - \cos\theta \sin^3\theta\, r^2 = -1 - \frac{\left(\sin 2\theta \sin^2\theta\right)\, r^2}{2} \\
&= -1 - \frac{\sin 2\theta (1 - \cos 2\theta) r^2}{4} < 0, \quad r < 2
\end{aligned}
$$

We see that for $r < 2$ we have $\theta' < 0$ and the trajectories tend to the origin going (non-uniformly) as spirals clockwise around the origin.

This example demonstrates the main idea with applications of the LaSalles invariance principle to asymptotic stability of equilibrium points.

Proposition. Simple version of applying LaSalle's invariance principle for asymptotic stability of equilibrium points by using "weak" Lyapunof functions.

(The complete version of LaSalle's invariance principle is Theorem 5.15. p. 183 that is considered a bit later)

We find a simple "weak" Lyapunov function $V_f(z) \leq 0$ for $z \in U$ in the domain $U \subset G$, $0 \in U$. This fact implies stability of the equilibrium. Then we check what happens on the set $V_f^{-1}(0)$ where $V_f(z) = 0$. If the set $V_f^{-1}(0)$ contains no other orbits except the equilibrium point, this equilibrium point in the origin must be asymptotically stable.

Any trajectory starting in a set $W \subset U$ that is positie invariant and compact will have positive orbit with compact closure. The set W can be chosen in this context as a subset $W \subset U$, bounded by a level set of the function V so that trajectories will not go outside W it because $V_f \leq 0$ in U. We need this property of trajectories in W for applying LaSalle's invariance principle describing ω- limit sets for positive orbits of solutions to ODEs.

Exercise.

Show that all trajectories of the system

$$\begin{aligned} x' &= y \\ y' &= -x - (1 - x^2)y \end{aligned}$$

that go through points in the domain $\left\| [x, y]^T \right\| < 1$, tend to the origin. Or by other words, show that the origin is an asymptotically stable equilibrium and that the circle $\left\| [x, y]^T \right\| < 1$ is it's domain of attraction.

Consider $V(x, y) = x^2 + y^2$.

$$\begin{aligned} V_f(x, y) &= 2xy - 2xy - (1 - x^2)y^2 = -(1 - x^2)y^2 \leq 0 \\ V_f^{-1}(0) &= \{(x, y) : y = 0\} \end{aligned}$$

The only invariant set is the origin $\{0\}$, therefore for trajectories starting in $\left\| [x, y]^T \right\| < 1$ the origin is an attractor and it is asymptotically stable with

$\left\| [x, y]^T \right\| < 1$ being the region (domain) of attraction.

More general formulation and a proof of the LaSalle's invariance principle use some general properties of transition mappings, and ω - limit sets. We collect them here and give some comments about their proofs.

We consider I.V.P. and corresponding transition mapping $\varphi(t, \xi)$ for the system

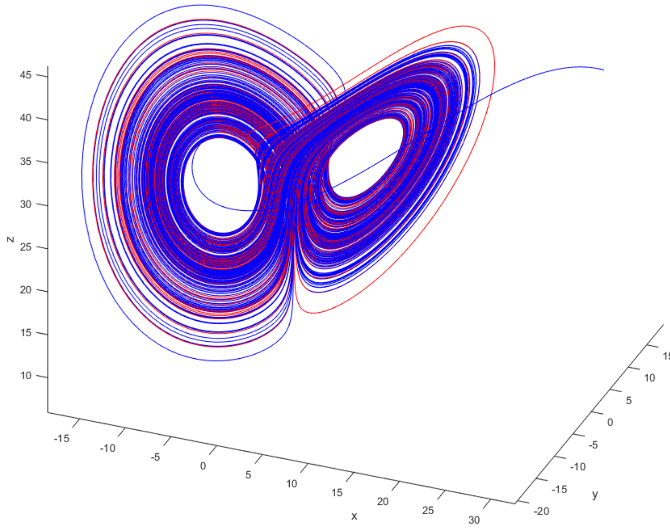
$$\begin{aligned} x' &= f(x) \\ x(0) &= \xi \end{aligned}$$

with $f : G \to \mathbb{R}^n$, G - open, $G \subset \mathbb{R}^n$, f is locally Lipschitz, $\xi \in G$.

1.1 Main theorem on the properties of limit sets.

The next theorem on the properties of ω - limit sets collects properties of ω - limit sets valid for systems of any dimension, in contrast with the Poincare - Bendixson theorem and it's generalization, that gives a description of ω - limit sets only for systems in plane, or on 2-dimensional manifolds.

Example. The Lorentz equation. Trajectory - blue, ω - limit set $\Omega(\xi)$ - red



$$\begin{aligned} x' &= -\sigma(x-y) \\ y' &= rx - y - xz \\ z' &= xy - bz \end{aligned}$$

A trajectory for $\sigma = 10$, $r = 28$, $b = 8/7$.

Main theorem about properties of ω - limit sets. Theorem 4.38, p.143

We keep the same limitations and notations for the autonomous system as above.

Let $\xi \in G$. Let the closure of the postive semi-orbit $O^+(\xi)$ be compact and contained in G,

Then $\mathbb{R}_+ \subset I_\xi$ and the ω - limit set $\Omega(\xi) \subset G$ is

1) non-empty

2) compact (bounded and closed)

3) connected

4) invariant (both positively and negatively) under the local flow $\varphi(t, \xi)$ generated by the ODE: namely for any ω - limit point $\eta \in \Omega(\xi)$, the maximal interval $I_\eta = \mathbb{R}$ for initial data in η, and $\varphi(t, \eta) \in \Omega(\xi)$ for all $t \in \mathbb{R}$.

5) $\varphi(t, \xi)$ approaches $\Omega(\xi)$ as $t \to \infty$:

$$\lim_{t \to \infty} \operatorname{dist}(\varphi(t, \xi), \Omega(\xi)) = 0$$

Remark

The most interesting statement in the theorem is statement 4). It means that ω - limit sets consist of orbits of solutions to the system. Taking a starting point η on the ω - limit set $\Omega(\xi)$ we get a trajectory $\varphi(t, \eta)$ that stays within this set $\Omega(\xi)$ infinitely long both in the future and in the past.

Remark

A simple tool to satisfy conditions in this theorem is to find a compact positively invariant set for the system, such that it contains the point ξ. It can be done using one of two methods discussed earlier.

Proofs of statements in the Theorem **4.38,** are based on the following mathematical tools:

1. general properties of compact sets for 1) ,2),
2. simple contradiction arguments and the definition of limit sets for 3)
3. and the transition property of the transition mapping $\varphi(t, \xi)$, together with continuity of $\varphi(t, \xi)$ for 4)
4. a contradiction argument togehter with the definition of ω - limit sets for 5).

We will only give a proof to 4) here supposing that 1), 2), and 3) are proven.

Proof to 4)

Let η be an ω - limit point for ξ: $\eta \in \Omega(\xi)$. By the definition there is a sequence of times $\{t_n\}$, $t_n \to \infty$ such that $\varphi(t_n, \xi) \to \eta$.

Consider the trajectory $\varphi(t, \eta)$ starting at η.

Denote by I_η corresponding maximal interval and consider **an arbitrary** $t \in I_\eta$, belonging to the maximal interval I_η.

We like to show that $\varphi(t, \eta) \in \Omega(\xi)$,namely that a trajectory starting in an ω - limit set $\Omega(\xi)$ stays within this ω - limit set forever in the future and in the past.

For n large enough $t + t_n \overset{def}{=} s_n \in \mathbb{R}_+$ - belongs to the maximal interval I_ξ of the solution $\varphi(t, \xi)$ for n large enough because $t_n \to \infty$ and $\mathbb{R}_+ \subset I_\xi$.

We are going to apply the group transition property for φ (similar to the Chapmen-Kolmogorov relation for linear systems) for the time interval: $t+t_n = s_n$

$$\varphi\left(s_n, \xi\right) = \varphi\left(t + t_n, \xi\right) = \varphi\left(t, \varphi(t_n, \xi)\right)$$

It is possible to apply because of the following argument.

The domain D of $\varphi(.,.)$ is open, $(t, \eta) \in D$, therefore there is a ball B around (t, η) such that $(t, \varphi(t_n, \xi)) \in B \subset D$ for n large enough because $\varphi(t_n, \xi) \to \eta$.

Therefore $t \in I_{\varphi(t_n, \xi)}$.

By continuity of φ it follows:

$$\varphi\left(s_n, \xi\right) = \varphi\left(t + t_n, \xi\right) = \varphi\left(t, \overset{\lim = \eta}{\varphi(t_n, \xi)}\right) \to \varphi(t, \eta), \quad n \to \infty$$

It means that $\varphi(t, \eta)$ is an ω - limit point for $\varphi\left(t, \xi\right)$ for any $t \in I_\eta$.

Moreoer, since $\Omega(\xi)$ is a compact subset in G, we obtain that $I_\eta = \mathbb{R}$ by

the Corollary 4.10 about the extension of an orbit that has compact closure in G.

■

LaSalle's invariance principle

We formulate now LaSalle's invariance principle that generalizes ideas that we discussed in the introductory example and gives a handy instrument for localizing ω - limit sets of non-linear systems in arbitrary dimension.

Theorem 5.12, p.180 (proof required at the exam)

Assume that f is locally Lipschitz $f : G \to \mathbb{R}^n$ as before and let $\varphi(t, \xi)$ denote the flow generated by the corresponding system

$$x' = f(x)$$

Let $U \subset G$ be non-empty and open. Let $V : U \to \mathbb{R}$ be continuously differentiable and such that $V_f(z) = \nabla V \cdot f(z) \leq 0$. for all $z \in U$. Let $\xi \in U$ be such that the closure of the semi-orbit $O^+(\xi)$ is compact and is contained in U,

i) then $\mathbb{R}_+ \subset I_\xi$ (maximal existence interval for ξ) and

ii) as $t \to \infty$, $\varphi(t, \xi)$ approaches the largest invariant set contained in $V_f^{-1}(0)$ that is the set where $V_f(z) = 0$.

Proof.

This proof given in the solution of Exercise 5.9, on p. 312.

Set $x(t) = \varphi(t, \xi)$. By continuity of V and compactness of the closure $cl(O^+(\xi))$, V is bounded on $O^+(\xi)$ and therefore the function $V(x(t))$ of time t is bounded.

• Since

$$\frac{d}{dt}(V(x(t))) = V_f(x(t)) \leq 0$$

for all $t \in \mathbb{R}_+$, $V(x(t))$ is non-increasing. We conclude that the limit $\lim_{t\to\infty} V(x(t))$ of the non-increasing function $V(x(t))$ must exist and is finite. We denote it by λ:

$$\lim_{t\to\infty} V(x(t)) = \lambda$$

• Take an arbitrary point $z \in \Omega(\xi)$ in the ω - limit set $\Omega(\xi)$. Then by the definition of ω - limit sets, there is a sequence $\{t_n\}$ in $\mathbb{R}_+$ such that

$\lim_{n\to\infty} t_n = \infty$ and

$$x(t_n) = \varphi(t_n, \xi) \longrightarrow z, \quad n \to \infty$$

We apply V to the left and right hand side in this limit calulation.

For any continuous function F and any convergent sequence $\{g_n\}$it is valid that

$$F(\lim_{n\to\infty} g_n) = \lim_{n\to\infty}(F(g_n))$$

• By the continuity of V it follows that $V(z) = \lim_{n\to\infty} V(x(t_n))$ and $\lim_{n\to\infty} V(x(t_n)) = \lim_{t\to\infty} V(x(t))$. Therefore

$$V(z) = \lim_{n\to\infty} V(x(t_n)) = \lim_{t\to\infty} V(x(t)) = \lambda.$$

This key point in the proof $\left(!!!\right)$ implies that for ALL z in the ω - limit set $\Omega(\xi)$ the test function V has the same value:

$$V(z) = \lambda, \quad \forall z \in \Omega(\xi) \tag{1}$$

• By the invariance of $\Omega(\xi)$ with respect to $\varphi(t, .)$,$\left(!!!\right)$ if $z \in \Omega(\xi)$, then $\varphi(t, z) \in \Omega(\xi)$ for all $t \in \mathbb{R}$.

(it is why the theorem is called the invariance principle!!!)

Therefore $V(\varphi(t, z)) = \lambda$ for all $t \in \mathbb{R}$ is a constant function of time t $\left(!!!\right)$.

A constant function must have zero derivative:

$$\frac{d}{dt} V(\varphi(t, z)) = V_f(\varphi(t, z)) = 0$$

for all $t \in \mathbb{R}_+$. Since $\varphi(0, z) = z$ and z is an arbitrary point in $\Omega(\xi)$ it follows that

$$V_f(z) = \left.\frac{d}{dt}V(\varphi(t,z))\right|_{t=0} = 0, \quad \forall z \in \Omega(\xi) \tag{2}$$

$$z = \varphi(0,z) \tag{3}$$

and therefore $\Omega(\xi) \subset V_f^{-1}(0)$,where $V_f^{-1}(0)$ is the set of x where $V_f(x) = 0$.

• The statement of the theorem follows now from the Main theorem about ω- limit sets (Theorem 4.38), that states: $\Omega(\xi)$ is an invariant set under the action of $\varphi(t,.)$, and $\varphi(t,\xi)$ apporaches $\Omega(\xi)$ as $t \to \infty$. It makes that $\varphi(t,\xi)$ must apporache the maximal invariant set inside $V_f^{-1}(0)$ that is easy to find by checking values f on the set $V_f^{-1}(0)$.

(The maximal invariant set contains invariant set $\Omega(\xi)$. Finding $\Omega(\xi)$ itself might be difficult).

Comment. It can be tempting to simplify the proof by concluding (1) from the fact that $(\nabla V)(z) = 0$ from all $z \in \Omega(\xi)$ which would imply (2).

However this conclusion is not valid, because the set $\Omega(\xi)$ is not open and therefore $V(z) = \lambda, \quad \forall z \in \Omega(\xi)$ does not imply $V_f(z) = 0, \quad \forall \in \Omega(\xi)$.

The invalidity of this conclusion is illustrated by the following simple example: $V(z) = \|z\|$, $\Omega(\xi) = \left\{z \in \mathbb{R}^N : \|z\| = 1\right\}$, then $V(z) = 1$ for all $z \in \Omega(\xi)$, but$(\nabla V)(z) = 2z \neq 0$ for all $z \in \Omega(\xi)$.

The following theorem follows rather directly from LaSalle's invariance principle and gives a practical criterium for asympototically stable equilibrium points using "weak" Lyapunov's functions.

Theorem 5.15. p. 183.

Let U be an open domain $U \subset G$, such that $0 \in U$ and a continuously differentiable function $V : U \to \mathbb{R}^n$ such that

$$V(0) = 0, \quad V(z) > 0, \forall z \in U \setminus \{0\}, \quad V_f(z) \leq 0, \forall z \in U \setminus \{0\}$$

and $\{0\}$ is the only invariant set contained in $V_f^{-1}(0)$. Then 0 is an asymptotically stable equilibrium.□

Proof follows from LaSalle's invariance principle and is a simple exercise.

Theorem 5.22, p. 188. On global asymtotic stability

Assume that $G = \mathbb{R}^n$. Let the hypothesis of the Theorem 5.15 hold with $U = G = \mathbb{R}^n$.

Namely for a continuously differential function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $V(z) > 0$ for all $z \in U\backslash\{0\}$, $V_f(z) \leq 0$ for all $z \in U\backslash\{0\}$, the origin $\{0\}$ is the only invariant set contained in $V_f^{-1}(0)$.

If in addition the Lyapunov function V is radially unbounded:

$$V(z) \to \infty, \quad \|z\| \to \infty$$

then the origin 0 is a globally stable equilibrium that means that all solutions $\|\varphi(t, \xi)\| \to 0$, as $t \to \infty$.

Exercise 5.17

The aim of this exercise is to show that the condition of radial unboundedness in Theorem 5.22 is essential.

Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(z) = f(z_1, z_2) = \begin{cases} (-z_1, z_2) & \text{if } z_1^2 z_2^2 \geq 1 \\ (-z_1, 2z_1^2 z_2^3 - z_2) & \text{if } z_1^2 z_2^2 < 1 \,. \end{cases}$$

Define $V: \mathbb{R}^2 \to \mathbb{R}$ by

$$V(z) = V(z_1, z_2) = z_1^2 + \frac{z_2^2}{1 + z_2^2} \,.$$

(a) Show that the equilibrium 0 of (5.1) is asymptotically stable.

(b) Show that the equilibrium 0 is *not* globally asymptotically stable.

(c) Show that V is not radially unbounded.

Examples of using La Salle's principle. Investigate stability of equilibrium points in the origin.

Example.

Consider the following system of ODEs: $\begin{cases} x' = & 2y \\ y' = & -x - (1-x^2)y \end{cases}$.

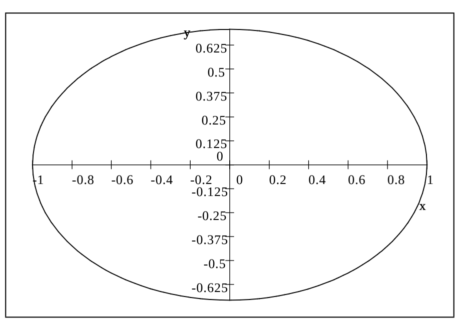
Show the asymptotic stability of the equilibrium point in the origin and find it's domain of attraction. **(4p)**

Solution.

We try the test function $V(x,y) = x^2 + 2y^2$ that leads to cancellation of mixed terms in the directional derivative V_f along trajectories. One can start with trying a more general test function $x^2 + ay^2$ with an arbitrary constant $a > 0$ and choose a so that indefinite terms in V_f would cancel.

$V_f(x,y) = 4xy - 4xy - 4y^2(1-x^2) = -4y^2(1-x^2)$ that is not positive for $|x| \le 1$. Therefore the origin is a stable stationary point. Checking the behavior of the system on the set of zeroes to $V_f(x,y)$ inside the stripe $|x| < 1$ we consider $(V_f)^{-1}(0) = \{(x,y) : y = 0, |x| < 1\}$. On this set $y' = -x$ and the only invariant set in $(V_f)^{-1}(0)$ is the origin. LaSalle's invariance principle implies that the origin is asymptotically stable.

The domain of attraction is the largest set bounded by a level set of $V(x,y) = x^2 + 2y^2$ inside the stripe $|x| \le 1$ where te monotonicity of the Lyapunov function V along trajectories is valid. The largest such set is the interior of the ellipse $x^2 + 2y^2 = C$ such that is touches the lines $x = \pm 1$. Taking points $(\pm 1, 0)$ we conclude that $1 = C$. and the boundary of the domain of attraction is the ellipse $x^2 + 2y^2 = 1$ with halfs of axes 1 and $\sqrt{0.5}$:



The next theorem gives a simple criterion for having the whole space as the domain of attraction for an asymptotically stable equilibrium point.

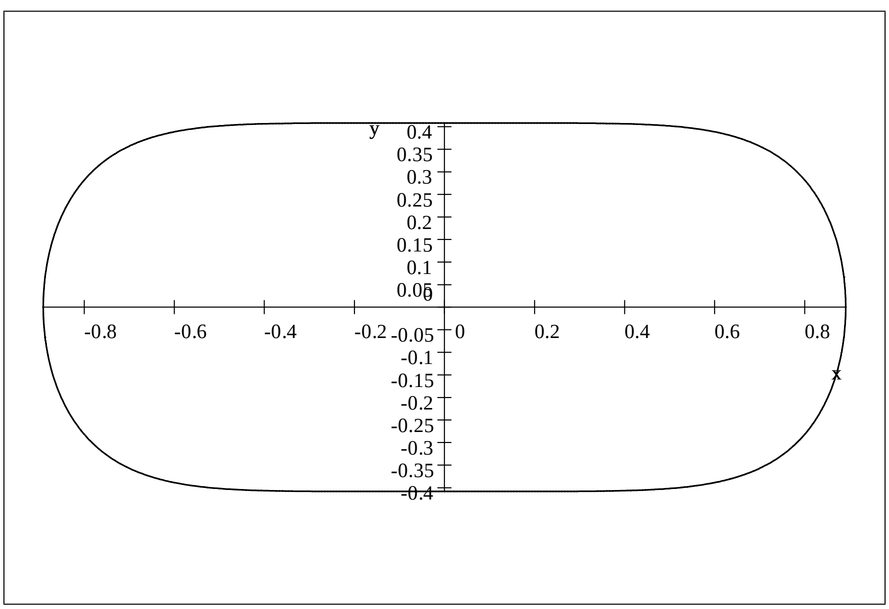
Example. Investigate stability of the equilibrium point in the origin.

$$\begin{aligned} x' &= -y - x^3 \\ y' &= x^5 \end{aligned}$$

We try our simplest choice of the Lyapunov function: $V(x, y) = x^2 + y^2$ and arrive to

$$V_f(x, y) = -2xy - 2x^4 + 2yx^5$$

It does not work because the expression $V_f(x, y)$ includes two indefinite terms: $2xy$ and $2yx^5$ that change sign around the origin. We try a more flexible expression by looking on particular expressions in the right hand side of the equation: $V(x, y) = x^6 + \alpha y^2$ where $\partial V/\partial x = 6x^5$ with the same power of x as in the equation, and the parameter α that can be adjusted later. V is a positive definite function: $V(0) = 0$ and $V(z) > 0$ for $z \neq 0$.The level sets to V look as flattened in y - direction ellipses. The curve $x^6 + 3y^2 = 0.5$ is depicted:



$$V_f(x,y) = 6x^5(-y-x^3) + 2\alpha y x^5 = -6x^5 y + 2\alpha x^5 y - 6x^8$$

We get again two indefinite terms, but they are proportional and the choice $\alpha = 3$ cancels them:

$$V_f(x,y) = -6x^8 \leq 0$$

Therefore the origin is a stable equilibrium point. $V_f(x,y) = 0$ on the whole $y-$axis that in our "general" theory is denoted by $V_f^{-1}(0)$.We check invariant sets of the system on the set $V_f^{-1}(0)$. We observe that $x' = -x^3$ (only this fact is important) and $y' = 0$ (it does not matter for $V_f^{-1}(0)$ that is $y-$axis). Therefore $\{0\}$ is the only invariant set on the y - axis. Trajectories starting on the y - axis go across it in all points except $\{0\}$. The LaSalle's invariance principle implies that all trajectories approach $\{0\}$ as t tends to infinity and the origin is asymptotically stable.

The test function $V(z) \to \infty$ as $\|z\| \to \infty$. It implies that the whole plain is a region or domain of attraction for the equilibrium point in the origin.■