Chalmers/GU
Mathematics sciences

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date:	19–10–31
Time:	8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Malin Nilsson, tel. 5325
Result announced:	19–11–11
	Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the problem (P) to:

$$
\begin{array}{lrl}
\text{maximize} & z = & x_1 + x_2, \\
\text{subject to} & & 2x_1 + x_2 \leq 2, \\
& & x_2 \leq 1, \\
& & -x_1 + x_2 \leq 1/2, \\
& & x_1 - x_2 \leq 1/2, \\
& & x_1, \quad x_2 \geq 0.
\end{array}
$$

(1p) a) Formulate the dual linear problem to (P) and convert the dual linear problem to standard form.

(1.5p) b) Solve the dual linear problem using phase I and phase II of the simplex method. Present an optimal solution to the dual linear problem or determine that no such exist.

(0.5p) c) Present an optimal solution to the original problem (P) or determine that no such exists. Utilize that the simplex algorithm computes the value of both primal- and dual variables.

(3p) Question 2

(unconstrained optimization)

Let $f(\boldsymbol{x}) := x_1^2 + 2x_1x_2 - 2x_2^2 + 4x_1$ and $\boldsymbol{x}^0 = (0,0)^{\mathrm{T}}$. Find the search directions at $\boldsymbol{x}^0$ for the following three unconstrained optimization methods:

a) The steepest descent method,

b) Newton's method,

c) Newton's method with the Levenberg–Marquardt modification using $\gamma = 8$ (where γ is the amount added to the diagonal of the Hessian).

In general, for which of the methods a)–c) are the directions found always descent directions? Motivate your answer.

(3p) Question 3

(Lagrangian relaxation)

Consider the problem to

$$\begin{array}{lrl} \text{minimize} & z = x_1 - 2x_2, & \\ \text{subject to} & x_1 - x_2 & \geq 2, \\ & x_1 + x_2 & \leq 5, \\ & x_1, x_2 & \geq 0. \end{array}$$

(2p) a) Lagrangian relax the first constraint. Use Lagrangian duality to obtain the optimal objective value z^*.

(1p) b) Use complementary slackness to obtain the optimal solution $\boldsymbol{x}^*$.

(3p) Question 4

(KKT conditions)

Consider the problem to

$$\begin{array}{rl} \text{minimize} & x_1x_2 + x_1x_3 + x_2x_3, \\ \text{subject to} & x_1 + x_2 + x_3 = 12. \end{array}$$

(2p) a) Write down the KKT conditions for the problem, and find all KKT points.

(1p) b) Does the problem have an optimal solution? Motivate!

(3p) Question 5

(modelling)

You are constructing a wooden product requiring N boards of lengths l_i, $i = 1, \dots, N$. Your local supplier currently has a stock of M boards with lengths L_j and at the prices p_j, $j = 1, \dots M$, where two boards of the same length and price are said to be of the same type. Let $\mathcal{S}_k \subset \{1, \dots, M\}$, denote all boards of type k and $d_k = p_j$, $j \in \mathcal{S}_k$, be the common price for board of type k, $k = 1, \dots, K$.

Moreover, each board bought can be cut, hence it can be enough for several boards of your wooden product. For example, if $L_1 = 3$, $l_1 = l_2 = 1$, then since $l_1 + l_2 \leq L_1$, the boards 1 and 2 of your wooden product can originate from board 1 in the stock.

The supplier also has an offer: every 4th board you buy of the same type, is for free.

Formulate an integer linear problem minimizing the cost of the boards purchased for your wooden product.

Question 6

(true or false)

Indicate for each of the following three statements whether it is true or false. Motivate your answers!

In each of the statements we consider $\boldsymbol{x}^* = (0, 0)^{\mathrm{T}}$ and the problem to:

$$
\begin{aligned}
\text{minimize} \quad & z = x_1, \\
\text{subject to} \quad & x_2^2 - x_1 x_2 + x_2 \leq 0, \\
& x_1, \quad x_2 \geq 0.
\end{aligned}
$$

(1p) a) *Claim:* $\boldsymbol{x}^*$ is a unique KKT Point to the problem.

(1p) b) *Claim:* Abadie's constraint qualification holds at $\boldsymbol{x}^*$.

(1p) c) *Claim:* The interior penalty method can converge to $\boldsymbol{x}^*$.

(3p) Question 7

(convergence of an exterior penalty method)

Let us consider a general optimization problem:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}), \\ \text{subject to} \quad & \boldsymbol{x} \in S, \end{aligned} \tag{1}$$

where $S \in \mathbb{R}^n$ is a non-empty, closed set and $f : \mathbb{R}^n \to \mathbb{R}$ is a given differentiable function. We assume that the feasible set S of the optimization problem (1) is given by the system of inequality and equality constraints:

$$S = \{\boldsymbol{x} \in \mathbb{R}^n \quad | \quad \begin{aligned} & g_i(\boldsymbol{x}) \leq 0, \quad i = 1, \dots, m, \\ & h_j(\boldsymbol{x}) = 0, \quad j = 1, \dots, \ell.\} \end{aligned} \tag{2}$$

where $g_i \in C^0$, $i = 1, \dots, m$, and $h_j \in C^0$, $j = 1, \dots, \ell$.

We choose a function $\Psi : \mathbb{R} \to \mathbb{R}_+$ such that $\Psi(s) = 0$ if and only if $s = 0$ (typical examples of Ψ are $\Psi(s) = |s|$, or $\Psi(s) = s^2$), and introduce the function

$$\nu \check{X}_S(\boldsymbol{x}) = \nu \left(\sum_{i=1}^{m} \Psi(\max\{0, g_i(\boldsymbol{x})\}) + \sum_{j=1}^{\ell} \Psi(h_j(\boldsymbol{x})) \right) \tag{3}$$

where the real number ν is called a *penalty parameter*.

We assume that for every $\nu > 0$ the approximating optimization problem to

$$\text{minimize} \quad f(\boldsymbol{x}) + \nu \check{X}_S(\boldsymbol{x}) \tag{4}$$

has at least one optimal solution $\boldsymbol{x}_\nu^*$.

Prove the following theorem.

THEOREM 1. *Assume that the original constrained problem (1) possesses optimal solutions. Then, every limit point of the sequence $\{\boldsymbol{x}_\nu\}$, $\nu \to +\infty$, of globally optimal solutions to (4) is globally optimal in the problem (1).*